

Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

_____ 1. Simplify the expression $(\sqrt{6} + \sqrt{24})^2$.

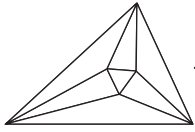
$a =$ _____ 2. The solution set of $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}} < \sqrt{8}$ can be written in the form $x > a$. Determine a exactly.

$(b, c) =$ _____ 3. Given that $\frac{6}{x^2-1} + \frac{b}{x-1} + \frac{c}{x+1} = \frac{4}{x+1}$, determine exactly the values of b and c .

_____ 4. For $0 < a < b$, if $a^3 = ka - 841$ and $b^3 = kb - 841$, determine exactly the value of $(ab)(a+b)$.

Name: _____

Team: _____



Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

54

1. Simplify the expression $(\sqrt{6} + \sqrt{24})^2$. $(\sqrt{6} + \sqrt{24})^2 = (\sqrt{6} + 2\sqrt{6})^2 = (3\sqrt{6})^2 = 54.$

$a = \frac{2}{25}$

2. The solution set of $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}} < \sqrt{8}$ can be written in the form $x > a$. Determine a exactly.

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}} < \sqrt{8} \Rightarrow \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{2}} + \sqrt{8} \Rightarrow \frac{1}{\sqrt{x}} < \frac{\sqrt{2}}{2} + 2\sqrt{2} \Rightarrow \frac{1}{\sqrt{x}} < \frac{5\sqrt{2}}{2} \Rightarrow \frac{1}{x} < \frac{25}{2} \Rightarrow x > \frac{2}{25}, \text{ so } a = \frac{2}{25}.$$

$(b, c) = (-3, 7)$

Graders: award
1 point for each
correct value.

3. Given that $\frac{6}{x^2-1} + \frac{b}{x-1} + \frac{c}{x+1} = \frac{4}{x+1}$, determine exactly the values of b and c .

Adding we obtain $\frac{6}{x^2-1} + \frac{b}{x-1} + \frac{c}{x+1} = \frac{6+b(x+1)+c(x-1)}{x^2-1} = \frac{(b+c)x+(b-c+6)}{(x-1)(x+1)}$. Therefore,

$$\frac{(b+c)x+(b-c+6)}{(x-1)(x+1)} = \frac{4}{x+1} \Rightarrow \frac{(b+c)x+(b-c+6)}{x-1} = \frac{4(x+1)}{x+1} \Rightarrow (b+c)x+(b-c+6) = 4(x-1) = 4x-4.$$

This gives the equations $b+c=4$ and $b-c+6=-4$. Adding these equations gives $2b+6=0$, so $b=-3$ and $c=7$.

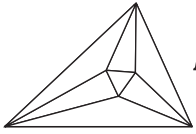
841

4. For $0 < a < b$, if $a^3 = ka - 841$ and $b^3 = kb - 841$, determine exactly the value of $(ab)(a+b)$.

Notice that $a^3 = ka - 841$ and $b^3 = kb - 841$ can both be rewritten as $a^3 - ka + 841 = 0$ and $b^3 - kb + 841 = 0$.

This means both a and b are solutions to the equation $x^3 - kx + 841 = 0$. Suppose c is the third solution. Using Vieta's formulas, we know $a + b + c = 0$ and $abc = -841$. Solving for c we get $c = -(a + b)$. Therefore,

$$abc = -841 = ab \cdot -(a+b) \Rightarrow ab(a+b) = 841.$$



Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

radius =

1. Determine exactly the radius of a circle in which the value of the circumference is one-third the value of its area.

DE =

2. In *Figure 2*, secant \overline{CD} and tangent \overline{CB} intersect circle A at points B and E . If $BC = 10$ and $CE = 5$, determine DE exactly.

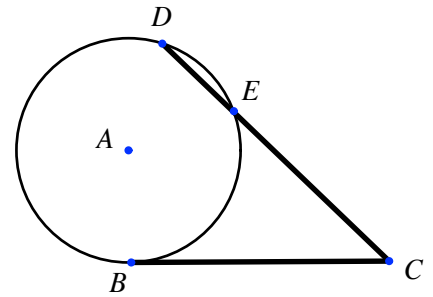


Figure 2

HD =

3. In circle D , chord \overline{FE} intersects the radius \overline{DG} at H , as shown in *Figure 3*. If $FH = 3$, $HE = 5$, and $HG = 1$, determine HD exactly.

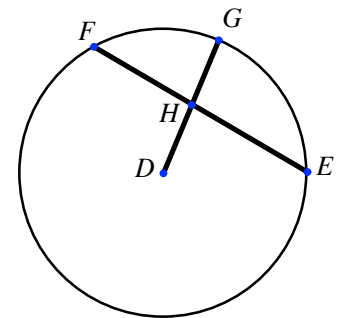


Figure 3

BE =

4. Triangle CBD is drawn so that side \overline{CB} runs through the center of circle A , \overline{BD} is tangent to the circle at B , and \overline{CD} intersects the circle at point E , as shown in *Figure 4*. If $AC = 4$ and $BD = 3$, determine BE exactly.

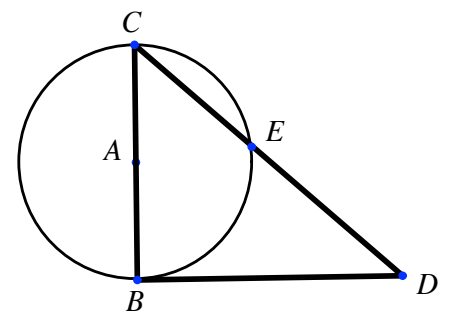
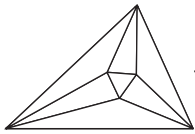


Figure 4

Name: _____

Team: _____



Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event B

SOLUTIONS

- 6 1. Determine exactly the radius of a circle in which the value of the circumference is one-third the value of its area.

$$2\pi r = \frac{1}{3}\pi r^2 \Rightarrow 6\cancel{\pi}r = \cancel{\pi}r^2 \Rightarrow r = 6$$

- DE = 15 2. In Figure 2, secant \overline{CD} and tangent \overline{CB} intersect circle A at points B and E. If $BC = 10$ and $CE = 5$, determine DE exactly.

Secant \overline{CD} and tangent \overline{CB} cut the circle with relationship that $CE \cdot CD = CB \cdot CB$. Therefore, $5 \cdot (5 + DE) = 10 \cdot 10 \Rightarrow 25 + 5DE = 100 \Rightarrow DE = 15$.

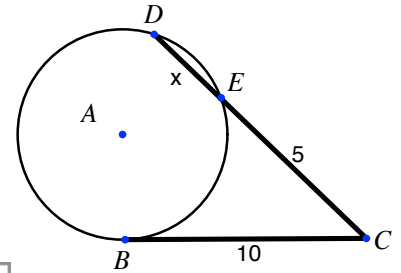


Figure 2

- HD = 7 3. In circle D, chord \overline{FE} intersects the radius \overline{DG} at H, as shown in Figure 3. If $FH = 3$, $HE = 5$, and $HG = 1$, determine HD exactly.

Extend radius \overline{DG} to create chord \overline{GI} . Let $HD = x$, which makes $DG = 1 + x$. Using the fact that the product of the parts of two intersecting chords in a circle are equal, we obtain $FH \cdot HE = IH \cdot HG \Rightarrow (1 + 2x) \cdot 1 = 3 \cdot 5 \Rightarrow x = HD = 7$.

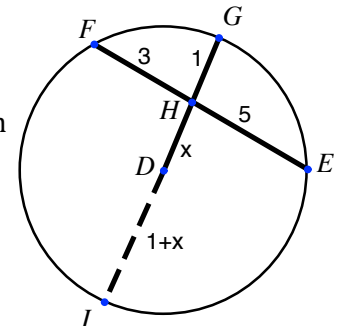


Figure 3

- BE = $\frac{24\sqrt{73}}{73}$ 4. Triangle CBD is drawn so that side \overline{CB} runs through the center of circle A, \overline{BD} is tangent to the circle at B, and \overline{CD} intersects the circle at point E, as shown in Figure 4. If $AC = 4$ and $BD = 3$, determine BE exactly.

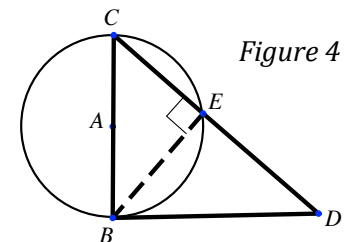


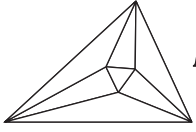
Figure 4

Let $CE = x$ and $DE = y$. Since \overline{BD} is tangent to the circle, triangle CBD is a right triangle. Using the Pythagorean Theorem, $3^2 + 8^2 = (x + y)^2 \Rightarrow \sqrt{73} = x + y$. Using the Power of a Point with respect to D and solving,

$$y(y + x) = 3^2 \Rightarrow y \cdot \sqrt{73} = 9 \Rightarrow y = \frac{9}{\sqrt{73}}. \text{ This means that } x = \frac{64}{\sqrt{73}}. \text{ Notice } \angle CEB \text{ must be right since it is}$$

inscribed and cuts the diameter. Using the Pythagorean Theorem again,

$$x^2 + (BE)^2 = 8^2 \Rightarrow \left(\frac{64}{\sqrt{73}}\right)^2 + (BE)^2 = 64 \Rightarrow BE = \frac{24\sqrt{73}}{73}.$$



Minnesota State High School Mathematics League

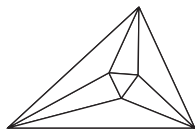
2016-17 Meet 4, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

- _____ 1. In arithmetic sequence $\{a_n\}$, $a_2 = 4$ and $a_{22} = 444$. Determine a_{222} exactly.
- _____ 2. In geometric sequence $\{a_n\}$, $a_1 = 2$ and $a_3 \cdot a_4 = 5$. Determine exactly $a_5 \cdot a_6 \cdot a_7$.
- _____ 3. What is the smallest integer n for which $1^3 + 2^3 + 3^3 + \dots + n^3$ is a multiple of 77.
- _____ 4. Determine the smallest $n > 2017$ for which $2 \cdot 3 \cdot 4 + 5 \cdot 6 \cdot 7 + 8 \cdot 9 \cdot 10 + \dots + (3n-1)(3n)(3n+1)$ is a multiple of 27.

Name: _____

Team: _____



Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event C

SOLUTIONS

4844

1. In arithmetic sequence $\{a_n\}$, $a_2 = 4$ and $a_{22} = 444$. Determine a_{222} exactly.

There are 20 common differences between 4 and 444. This means the common difference is $\frac{444-4}{22-2} = 22$, so $a_{222} = 444 + 200 \cdot 22 = 4844$.

$\frac{125}{8}$

or

$15\frac{5}{8}$

2. In geometric sequence $\{a_n\}$, $a_1 = 2$ and $a_3 \cdot a_4 = 5$. Determine exactly $a_5 \cdot a_6 \cdot a_7$.

or

15.625

Let the ratio of the geometric sequence be r . Then $a_3 \cdot a_4 = 2r^2 \cdot 2r^3 = 5 \Rightarrow r^5 = \frac{5}{4}$.
Therefore, $a_5 \cdot a_6 \cdot a_7 = 2r^4 \cdot 2r^5 \cdot 2r^6 = 8r^5 \cdot r^5 \cdot r^5 = 8 \cdot \left(\frac{5}{4}\right)^3 = \frac{125}{8}$.

21

3. What is the smallest integer n for which $1^3 + 2^3 + 3^3 + \dots + n^3$ is a multiple of 77.

Using the formula for the sum of cubes, the left side of the equation is $\frac{n^2(n+1)^2}{4}$. In order to be a multiple of 77, the numerator must be a multiple of 77. To be a multiple of 77 the number must be divisible by both 7 and by 11. The values of n for which the numerator is divisible by 11 are 10, 11, 21, 22, 32, 33, The first number in this list for which either n or $n + 1$ is a multiple of 7 is $n = 21$.

2024

4. Determine the smallest $n > 2017$ for which $2 \cdot 3 \cdot 4 + 5 \cdot 6 \cdot 7 + 8 \cdot 9 \cdot 10 + \dots + (3n-1)(3n)(3n+1)$ is a multiple of 27.

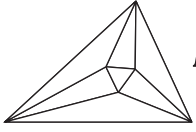
The sum can be written as

$$2 \cdot 3 \cdot 4 + 5 \cdot 6 \cdot 7 + 8 \cdot 9 \cdot 10 + \dots + (3n-1)(3n)(3n+1) = \sum_{i=1}^n (3i-1)(3i)(3i+1) = \sum_{i=1}^n 27i^3 - 3i = 27 \sum_{i=1}^n i^3 - 3 \sum_{i=1}^n i$$

Substituting the sum of cubes and sum of integers formulas in we obtain

$$27 \sum_{i=1}^n i^3 - 3 \sum_{i=1}^n i = \frac{27n^2(n+1)^2}{4} - \frac{3n(n+1)}{2} = \frac{3n(n+1)(9n^2+9n-2)}{4}$$

For this to be a multiple of 27, we need either n or $n + 1$ to be a multiple of 9. The quadratic term will never be divisible by 9. The smallest $n > 2017$ for which this occurs is 2024.



Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

(x, y) = _____

1. Determine exactly the vertex of $y = 2x^2 + 4x + 1$.

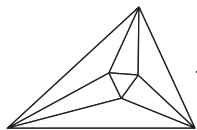
2. An ellipse has center $(-2, 7)$ and is tangent to both the x and y -axes. If the minor axis is parallel to the x -axis, determine exactly the sum of the lengths of the major and minor axes.

3. Write an equation of the parabola with directrix $x = 4$ and focus at the origin.

4. An isosceles triangle is constructed inside the ellipse $x^2 + 17y^2 - 4x - 68y + 4 = 0$. It has a vertex at each focus and the third vertex at a co-vertex (vertex on the minor axis) of the ellipse. Determine exactly the area of the isosceles triangle.

Name: _____

Team: _____



Minnesota State High School Mathematics League

2016-17 Meet 4, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

$$(x, y) = \boxed{(-1, -1)}$$

1. Determine exactly the vertex of $y = 2x^2 + 4x + 1$.

The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. Therefore, $x = \frac{-b}{2a} = \frac{-4}{2 \cdot 2} = -1$. Substituting this into the original equation gives a y value of -1 , so the vertex is $(-1, -1)$.

18

2. An ellipse has center $(-2, 7)$ and is tangent to both the x and y -axes. If the minor axis is parallel to the x -axis, determine exactly the sum of the lengths of the major and minor axes.

Since the ellipse is tangent to the x and y -axis and the center of the ellipse is $(-2, 7)$, the distance the center is from each axis will be half the major and minor axis of the ellipse. Therefore, the length of the major axis is 14 and the length of the minor axis is 4, which gives a sum of 18.

$$\boxed{(y-0)^2 = -8(x-2)}$$

or
any equivalent
form

3. Write an equation of the parabola with directrix $x = 4$ and focus at the origin.

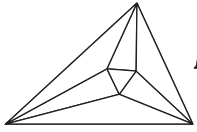
The parabola will open to the left and is symmetric to the x -axis since its directrix is $x = 4$ and its focus is the origin. The vertex is located directly between the directrix and the origin, namely $(2, 0)$. A parabola with the horizontal axis of symmetry can be written in the form $(y - k)^2 = 4p(x - h)$ where (h, k) is the vertex and p is the directed distance from the directrix. Substituting we obtain the parabola $(y - 0)^2 = -8(x - 2)$.

16

4. An isosceles triangle is constructed inside the ellipse $x^2 + 17y^2 - 4x - 68y + 4 = 0$. It has a vertex at each focus and the third vertex at a co-vertex (vertex on the minor axis) of the ellipse. Determine exactly the area of the isosceles triangle.

Rewriting the equation in standard form by completing the square gives

$$x^2 - 4x + 4 + 17y^2 - 68y + 68 - 68 = 0 \Rightarrow \frac{(x-2)^2}{68} + \frac{(y-2)^2}{4} = 1.$$
 Using $c^2 = a^2 - b^2 = 68 - 4 = 64$, the length of the focus is $c = 8$. Therefore, the area of the triangle is $\frac{1}{2}(2c)b = \frac{1}{2} \cdot (2 \cdot 8) \cdot 2 = 16$.



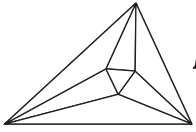
Minnesota State High School Mathematics League

2016-17 Meet 4, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

- _____ 1. Find the *sum* of all positive integer values k such that the solutions to $3x^2 + kx - 7 = 0$ are rational.
- _____ 2. A sphere has an inscribed rectangular prism whose dimensions follow a 4:1:1 ratio. Determine exactly the ratio of the volume of the prism to the volume of the sphere.
- _____ 3. The ellipse $7x^2 + 13y^2 - 6xy\sqrt{3} - 144 = 0$ has a circle inscribed inside it and a second circle circumscribing it. If both circles have the same center as the ellipse, determine exactly the difference between the areas of the larger and the smaller circle.
- _____ 4. Determine the smallest possible value of $a + b + c$, if a , b , and c are positive integers for which $\frac{a!}{b!} + \sum_{k=1}^c \binom{c}{k} = 2016$.
- _____ 5. Let b be a positive integer greater than 1. Determine all values of b such that there are exactly 306 two-digit numbers with a base of b .
- (x, y) = _____ 6. Determine exactly the focal point of $xy = 3$ that lies in the third quadrant.

Team: _____



Minnesota State High School Mathematics League

2016-17 Meet 4, Team Event

SOLUTIONS (page 1)

24

1. Find the *sum* of all positive integer values k such that the solutions to $3x^2 + kx - 7 = 0$ are rational.

$\frac{2\sqrt{2}}{9\pi}$

2. A sphere has an inscribed rectangular prism whose dimensions follow a 4:1:1 ratio. Determine exactly the ratio of the volume of the prism to the volume of the sphere.

27 π

3. The ellipse $7x^2 + 13y^2 - 6xy\sqrt{3} - 144 = 0$ has a circle inscribed inside it and a second circle circumscribing it. If both circles have the same center as the ellipse, determine exactly the difference between the areas of the larger and the smaller circle.

1995

4. Determine the smallest possible value of $a + b + c$, if a , b , and c are positive integers for

$$\text{which } \frac{a!}{b!} + \sum_{k=1}^c \binom{c}{k} = 2016.$$

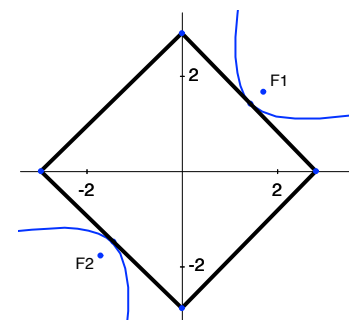
18

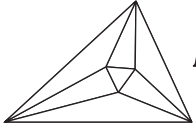
5. Let b be a positive integer greater than 1. Determine all values of b such that there are exactly 306 two-digit numbers with a base of b .

$(x, y) = (-\sqrt{6}, -\sqrt{6})$ 6.

- Determine exactly the focal point of $xy = 3$ that lies in the third quadrant.

Figure 6





Minnesota State High School Mathematics League

2016-17 Meet 4, Team Event

SOLUTIONS (page 2)

- Using the Quadratic Formula we get $x = \frac{-k \pm \sqrt{k^2 - 4 \cdot 3 \cdot -7}}{2 \cdot 3} = \frac{-k \pm \sqrt{k^2 + 84}}{6}$. For a positive integer m , set $k^2 + 84 = m^2 \Rightarrow m^2 - k^2 = 84$. Factoring gives $(m + k)(m - k) = 84$. Since both k and m are positive integers, $m + k$ must be the larger factor and $m - k$ the smaller. This gives us the following possibilities for $(m + k, m - k)$: $(84, 1)$, $(42, 2)$, $(28, 3)$, $(21, 4)$, $(14, 6)$, and $(12, 7)$. Since m and k are integers, $m + k$ and $m - k$ must have the same parity, so our only possibilities are $(42, 2)$ and $(14, 6)$. These give $(m, k) = (22, 20)$ and $(10, 4)$ respectively. Therefore, the sum of all k is $\boxed{24}$.
- Let the square base of the prism have side length x . Since the ratio of the sides of the prism is $4:1:1$, the height of the prism would be $4x$. The prism and the sphere share same center and the longest diagonal of the prism will go through the center of the sphere. The length of that diagonal will be $\sqrt{x^2 + x^2 + (4x)^2} = 3x\sqrt{2}$. The radius of the sphere is half that, so the ratio is $\frac{x \cdot x \cdot 4x}{\frac{4}{3} \cdot \pi \cdot \left(\frac{3x\sqrt{2}}{2}\right)^3} = \frac{2\sqrt{2}}{9\pi}$.
- First find the angle of rotation by using $\cot 2\theta = \frac{7-13}{-6\sqrt{3}} = \frac{\sqrt{3}}{3}$, so $\theta = \frac{\pi}{6}$. Using the substitutions for rotation $x = x' \cos \theta - y' \sin \theta$ and $y = x' \sin \theta + y' \cos \theta$ to rewrite the equation of an ellipse gives $7 \left(x' \cdot \frac{\sqrt{3}}{2} - y' \cdot \frac{1}{2} \right)^2 + 13 \left(x' \cdot \frac{1}{2} + y' \cdot \frac{\sqrt{3}}{2} \right)^2 - 6\sqrt{3} \left(x' \cdot \frac{\sqrt{3}}{2} - y' \cdot \frac{1}{2} \right) \left(x' \cdot \frac{1}{2} + y' \cdot \frac{\sqrt{3}}{2} \right) - 144 = 0$. This simplifies to $(4x'^2 + 16y'^2) - 144 = 0 \Rightarrow \frac{x'^2}{36} + \frac{y'^2}{9} = 1$. The inscribed circle's radius is the semi-minor axis of 3 and the circumscribed circle's radius is the semi-major axis of 6. The difference in areas is $36\pi - 9\pi = \boxed{27\pi}$.
- It is important to note that the summation is equal to $2^c - 1$, which is odd, so the fraction $\frac{a!}{b!}$ must also be odd. This can be odd only when $b = a$ or $b = a - 1$. If $b = a$, then $2^c - 1 = 2015$. However, solving this would not give an integer value for c . So we must have $b = a - 1$, which simplifies to $a + 2^c - 1 = 2016 \Rightarrow 2^c + a = 2017$. To get the smallest value of $a + b + c$, we should seek to minimize a and maximize c . This is achieved for $c = 10$, so $a = 2017 - 2^{10} = 993$. The smallest sum is then $993 + 992 + 10 = \boxed{1995}$.
- The smallest two digit number would be 10_b while the largest would be $(b-1)(b-1)_b$. Thus, $(b-1) \cdot b + (b-1) - (1 \cdot b + 0) + 1 = 306 \Rightarrow b^2 - b = 306 \Rightarrow b^2 - b - 306 = 0 \Rightarrow (b-18)(b+17) = 0$. So b is $\boxed{18}$.
- The curve $xy = 3$ is a hyperbola rotated 45° . The distance between vertices $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$ is $2\sqrt{6}$. The other side of the rectangle is formed by tangents to the hyperbola that also have a length $2\sqrt{6}$, as shown in Figure 6. If the hyperbola were of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, both a and b would be $\sqrt{6}$. Thus $c^2 = a^2 + b^2 \Rightarrow c^2 = 12 \Rightarrow c = 2\sqrt{3}$. If that segment length were rotated 45° clockwise, the focal point would occur at $\boxed{(-\sqrt{6}, -\sqrt{6})}$.