Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

________________ 1. Determine exactly the area of the region in the first quadrant bounded by $\frac{x}{4}+\frac{y}{10}=1$.

________________ 2. Given $\begin{vmatrix} 2 & 9 \\ 3 & b \end{vmatrix} = 2$, determine exactly $\begin{vmatrix} 9 & 2 \\ b & 3 \end{vmatrix}$.

________________ 3. Five years ago I was the age my brother is now. When I am fifty, my brother will be three less than twice the age he is now. How old am I?

________________ 4. Find the sum of all positive integers a and b where $a > b$ and the determinant

$$\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{vmatrix} = -13.$$

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event A

SOLUTIONS

20

1. Determine exactly the area of the region in the first quadrant bounded by $\frac{x}{4}+\frac{y}{10}=1$.

The x and y intercepts are (4, 0) and (0, 10) making a right triangle shown in Figure 1 whose area is $\frac{1}{2}\bullet 4\bullet 10=20$.

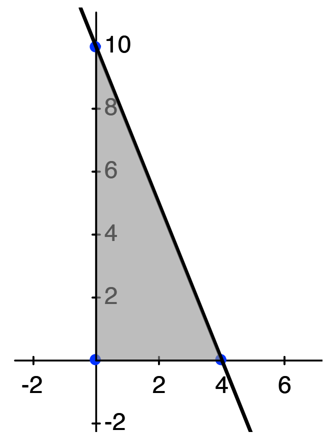


Figure 1

-2

2. Given $\begin{vmatrix} 2 & 9 \\ 3 & b \end{vmatrix}=2$, determine exactly $\begin{vmatrix} 9 & 2 \\ b & 3 \end{vmatrix}$.

Since $\begin{vmatrix} 2 & 9 \\ 3 & b \end{vmatrix}=2$, *2b – 27 = 2* $\Rightarrow b=\frac{29}{2}$. *Substituting this into our other matrix and solving for the determinant gives* $27-2\left(\frac{29}{2}\right)=-2$. ***Note: swapping columns of a 2x2 matrix flips the sign of the determinant***

29

3. Five years ago I was the age my brother is now. When I am fifty, my brother will be three less than twice the age he is now. How old am I?

Let x be the age of my brother now. My age now is then x + 5. When I am fifty, my younger brother will be 45. So, $2x-3=45 \Rightarrow x=24$. *Therefore, I am 29 years old.*

8

4. Find the <u>sum</u> of all positive integers a and b where $a > b$ and the determinant $\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{vmatrix}=-13$.

The determinant of $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{bmatrix}=-13$ *when*

$1(1\bullet 1-a\bullet b)-0(0\bullet 1-a\bullet b)+a(0\bullet b-b\bullet 1)=-13 \Rightarrow 1-2ab=-13 \Rightarrow ab=7$. *Since 7 is prime and a > b, a = 7 and b = 1, so a + b is 8.*

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

______________ 1. Determine exactly the surface area of a sphere whose volume is 36π.

______________ 2. When the side lengths of a cube are all increased by 1, the surface area increases by 90. Calculate the volume of the original cube.

AC : BD = ______________ 3. Points *A*, *B*, *C*, and *D* are located on circle *O* as shown in *Figure* 3. If chords *AB* = *BC* = 3*CD* = 3*AD*, determine exactly the ratio *AC* : *BD*.

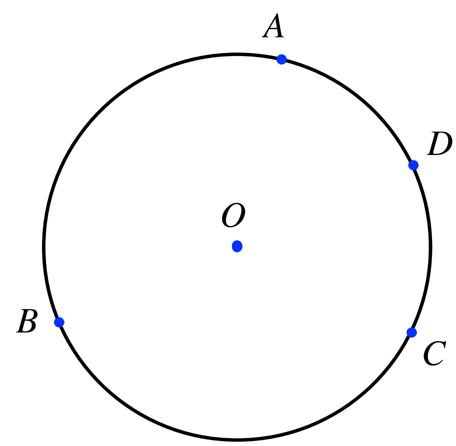


Figure 3

______________ 4. Shown in Figure 4 is a pyramid which has a square base with side lengths of 6, two sides of length 13, and two sides of length 11. Calculate the volume of this pyramid.

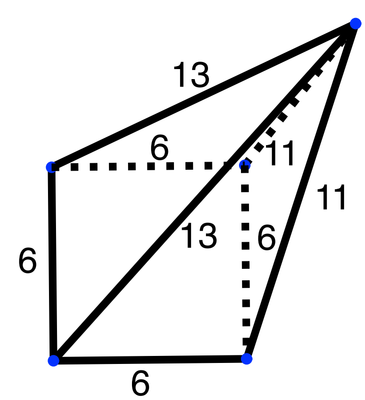


Figure 4

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event B

SOLUTIONS

$\boxed{36\pi}$ 1. Determine exactly the surface area of a sphere whose volume is 36π.

> $36\pi = \frac{4}{3}\pi r^3 \Rightarrow 27 = r^3 \Rightarrow r = 3.$ *Therefore, the surface area is* $4\pi r^2 = 4\pi \bullet 3^2 = 36\pi.$

$\boxed{343}$ 2. When the side lengths of a cube are all increased by 1, the surface area increases by 90. Calculate the volume of the original cube.

> *Let the side length of the original cube be a. The surface area of the new cube is* $6(a+1)^2$ *and surface area of the original cube is* $6a^2$*, and since the surface area increases by 90,* $6(a+1)^2 - 6a^2 = 90 \Rightarrow 12a + 6 = 90 \Rightarrow a = 7$*. Therefore, the volume of the original cube is* $7^3 = 343$*.*

$\boxed{\frac{3}{5}}$ 3. Points A, B, C, and D are located on circle O as shown in *Figure* 3. If chords $AB = BC = 3CD = 3AD$, determine exactly the ratio $AC : BD$.

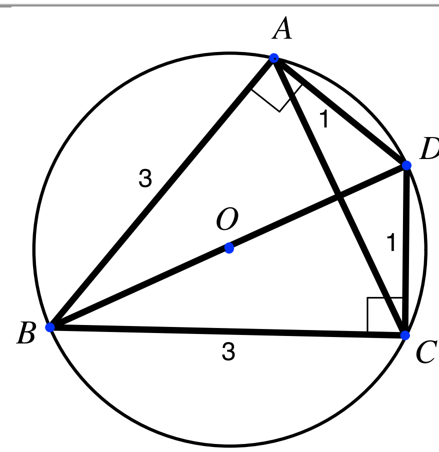


Figure 3

> *Let AB, BC, CD, and AD be 3, 3, 1, and 1 respectively. Triangles ABD and CBD are congruent, and since ABCD is a cyclic quadrilateral, those triangles are right. Using the Pythagorean Theorem, BD =* $\sqrt{1^2+3^2} = \sqrt{10}$*. Applying Ptolemy's Theorem,* $\sqrt{10} \bullet AC = 1 \bullet 3 + 1 \bullet 3 \Rightarrow AC = \frac{6}{\sqrt{10}}$*. Thus,* $\frac{AC}{BD} = \frac{\frac{6}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{5}$*.*

AC : BD = $\boxed{12\sqrt{111}}$ ***or*** $\boxed{126.427}$ ***or*** $\boxed{126.428}$

4. Shown in Figure 4 is a pyramid which has a square base with side lengths of 6, two sides of length 13, and two sides of length 11. Calculate the volume of this pyramid.

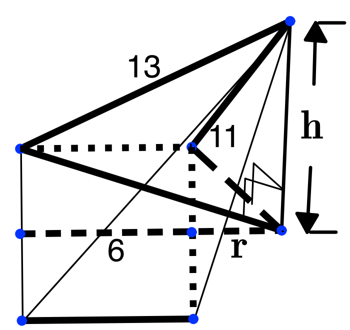


Figure 4

> *Let h be the height of the pyramid. The point in the base directly below the apex of the pyramid is r units from one of the sides of the base, 6 + r units from the opposite side, and 3 units from the other pair of sides as shown in Figure 4. Using the Pythagorean Theorem on the two right triangles shown in Figure 4, we get* $(6+r)^2 + 3^2 + h^2 = 13^2$ *and* $r^2 + 3^2 + h^2 = 11^2$*. Subtracting these equations gives 12r + 36 = 48* $\Rightarrow$ *r = 1. Therefore,* $h^2 = 111^2 \Rightarrow h = \sqrt{111}$*, which means the volume is* $\frac{1}{3} \bullet 6^2 \bullet \sqrt{111} = 12\sqrt{111}.$

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________ 1. Determine exactly $\cos^{-1}\left(\frac{-1}{2}\right)$.

(x , y) = ________ 2. Write as an ordered pair (x, y) the point where y = 3x – 4 intersects its inverse.

AC = ________ 3. In ΔABC, $AB = 20\sqrt{3}$, $m\angle CAB = 45°$, $m\angle ACB = 60°$. Determine exactly *AC.*

________ 4. In isosceles ΔMNP, *MN* = *MP*, *NP* = 1, and cos *M* + cos *N* + cos *P* = 1.18. If all sides have integer lengths, determine exactly the perimeter of ΔMNP.

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\frac{2\pi}{3}}$ or $\boxed{120°}$

1. Determine exactly $\cos^{-1}\left(\frac{-1}{2}\right)$.

The inverse cosine function has a range $0<\cos^{-1}(x)<\pi$*, so we are looking for the angle whose cosine is* $\frac{-1}{2}$*. Since* $\cos\left(\frac{2\pi}{3}\right)=\frac{-1}{2}$*, the angle we seek is* $\frac{2\pi}{3}$.

(x, y) = $\boxed{(2,2)}$

Graders: award 1 point for each correct coordinate.

2. Write as an ordered pair (x, y) the point where y = 3x – 4 intersects its inverse.

The function y = 3x – 4 and its inverse are reflections of each other over the line y = x. This means y = 3x – 4 will intersect its inverse when y = x. Substituting the y for an x gives x = 3x – 4 ⇒ x = 2. Since y = x, y is also 2 giving the ordered pair (2,2).

AC = $\boxed{10\left(\sqrt{6}+\sqrt{2}\right)}$ or $\boxed{20\left(\sqrt{2+\sqrt{3}}\right)}$

3. In ΔABC, $AB=20\sqrt{3}$, $m\angle CAB=45°$, $m\angle ACB=60°$. Determine exactly *AC*.

Using the Law of Sines gives $\frac{20\sqrt{3}}{\sin 60°}=\frac{AC}{\sin 75°}$*. First, apply the angle addition property for sine to get*

$\sin 75°=\sin(45°+30°)=\sin 45°\bullet\cos 30°+\sin 30°\bullet\cos 45°=\frac{\sqrt{6}+\sqrt{2}}{4}$*. Therefore, AC =* $\frac{20\sqrt{3}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)}{\frac{\sqrt{3}}{2}}=10\left(\sqrt{6}+\sqrt{2}\right)$.

$\boxed{11}$

4. In isosceles ΔMNP, *MN* = *MP*, *NP* = 1, and cos *M* + cos *N* + cos *P* = 1.18. If all sides have integer lengths, determine exactly the <u>perimeter</u> of ΔMNP.

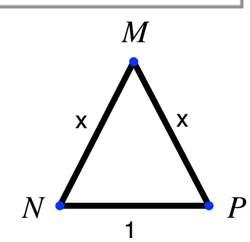


Figure 4

Let $MN=MP=x$ *as shown in Figure 4. By the Law of Cosines,* $x^2=1^2+x^2-2\cdot 1\cdot x\cdot\cos N\Rightarrow\cos N=\frac{1}{2x}=\cos P$.

Applying the Law of Cosines again gives $1^2=x^2+x^2-2\cdot x\cdot x\cdot\cos M\Rightarrow\cos M=\frac{2x^2-1}{2x^2}$*. Substituting gives*

$$\cos M+\cos N+\cos P=\frac{2x^2-1}{2x^2}+\frac{1}{2x}+\frac{1}{2x}=1.18\Rightarrow 2x^2-1+2x=2.36x^2\Rightarrow 0.36x^2-2x+1=0\Rightarrow 9x^2-50x+25=0\Rightarrow(9x-5)(x-5)=0.$$

Since the lengths of the sides are integers, x = 5 and the perimeter of ΔMNP *is 11.*

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

______________ 1. For what x value will $4\log_3 x = 4$?

$a + b =$ ______________ 2. The expression $\dfrac{3\left(3^{2n-1}\right)+9^{n-1}}{27^{\frac{2n}{3}-1}}$ can be written in the form $3^a + 3^b$. Determine exactly the sum $a + b$.

$\dfrac{x}{y} =$ ______________ 3. If $x > 2y > 0$ and $2\log\left(x-2y\right) = \log x + \log y$, determine $\dfrac{x}{y}$ exactly.

$a =$ ______________

$b =$ ______________ 4. The solution set of all x values for which $\log_4 x - \log_4 2 + \log_4\left(x-4\right) \le 2$ can be written as $a < x \le b$. Determine exactly the values of a and b.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2016-17 Meet 3, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

[3] 1. For what x value will $4\log_3 x = 4$?

$4\log_3 x = 4 \Rightarrow \log_3 x = 1 \Rightarrow x = 3^1 = 3.$

$a + b =$ [4] 2. The expression $\dfrac{3\left(3^{2n-1}\right)+9^{n-1}}{27^{\frac{2n}{3}-1}}$ can be written in the form $3^a + 3^b$. Determine exactly the sum $a + b$.

$\dfrac{3\left(3^{2n-1}\right)+9^{n-1}}{27^{\frac{2n}{3}-1}} = \dfrac{3^{2n}+3^{2(n-1)}}{3^{3\left(\frac{2}{3}n-1\right)}} = \dfrac{3^{2n}+3^{2n-2}}{3^{2n-3}} = \dfrac{3^{2n}}{3^{2n-3}} + \dfrac{3^{2n-2}}{3^{2n-3}} = 3^3 + 3^1$. *Therefore, a + b = 4.*

$\dfrac{x}{y} =$ [4] 3. If $x > 2y > 0$ and $2\log(x-2y) = \log x + \log y$, determine $\dfrac{x}{y}$ exactly.

Using the product and power properties of logarithms, we can write our given equation as $\log(x-2y)^2 = \log(xy)$.

Equating the powers gives $(x-2y)^2 = xy \Rightarrow x^2 - 5xy + 4y^2 = 0 \Rightarrow (x-4y)(x-y) = 0.$

Since $x > 2y$, $x - 4y = 0 \Rightarrow x = 4y \Rightarrow \dfrac{x}{y} = \dfrac{4y}{y} = 4.$

$a =$ [4] 4. The solution set of all x values for which $\log_4 x - \log_4 2 + \log_4(x-4) \le 2$ can be written as $a < x \le b$. Determine exactly the values of a and b.

$b =$ [8]

Graders: award 1 point for each correct response.

First, the domain of the lefthand side of the inequality is $x > 4$. *Applying the quotient and product properties of logarithms we obtain* $\log_4 x - \log_4 2 + \log_4(x-4) \le 2 \Rightarrow \log_4 \dfrac{x}{2}(x-4) \le 2$. *Rewriting the inequality gives* $\dfrac{x}{2}(x-4) \le 16 \Rightarrow \dfrac{x^2}{2} - 2x \le 16 \Rightarrow x^2 - 4x - 32 \le 0 \Rightarrow (x+4)(x-8) \le 0$. *The solution to this inequality is* $-4 \le x \le 8$. *However, our domain is restricted to values above 4. Therefore, our solution set is* $4 < x \le 8$, *which means a = 4 and b = 8.*

Minnesota State High School Mathematics League

2016-17 Meet 3, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

$(x, y, z) =$ __________

1. The positive integers x, y, and z sum to 20, with $x > y > z$. The sum of the positive differences of the three possible pairs of integers is 18. None of the integers is prime. Write the integers as an ordered triple (x, y, z).

$AE =$ __________

2. In *Figure 2*, C is the intersection of $\overline{AD}$ and $\overline{BE}$. If $AB = BC$, $AD = 2\sqrt{6} - \sqrt{2}$, $CD = \sqrt{6}$, $m\angle CAB = 15°$, and $m\angle CDE = 120°$, calculate AE.

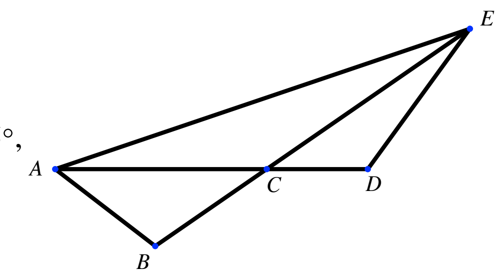


Figure 2

____

3. Determine the set of all real numbers x such that $\sqrt{14-x} - \sqrt{x-1} \le 1$.

____

4. $z^{18} = 1$ has 18 solutions in the complex plane of the form $r(\cos\theta + i\sin\theta)$. Let z_1 be the solution with the smallest positive angle, z_2 be the next largest angle, and so on. When graphed in the complex plane, the points z_5, z_7, and z_{11} are the vertices of a triangle. Calculate the length of the segment from z_5 to z_{11}.

Area = __________

5. In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$, and the diagonals intersect at P. If $AB = AP = DP = \sqrt{CP} = 3$, determine exactly the area of $ABCD$.

____

6. What is the area bounded by $y = 3x$, $y = \frac{1}{4}x$, $y = 3x - 11$, and $y = \frac{1}{4}x + 11$?

Team: ____________________

Minnesota State High School Mathematics League

2016-17 Meet 3, Team Event

SOLUTIONS (page 1)

(x, y, z) = $\boxed{(10,9,1)}$

1. The positive integers x, y, and z sum to 20, with $x > y > z$. The sum of the positive differences of the three possible pairs of integers is 18. None of the integers is prime. Write the integers as an ordered triple (x, y, z).

AE = $\boxed{4.009}$ **or** $\boxed{4.008}$

2. In *Figure 2*, C is the intersection of $\overline{AD}$ and $\overline{BE}$. If $AB = BC$, $AD = 2\sqrt{6} - \sqrt{2}$, $CD = \sqrt{6}$, $m\angle CAB = 15°$, and $m\angle CDE = 120°$, calculate AE.

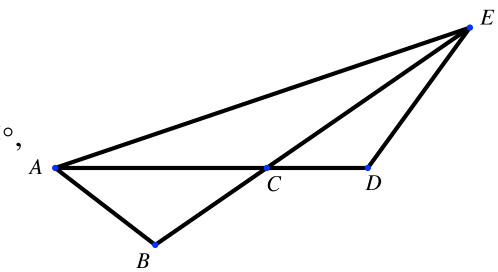


Figure 2

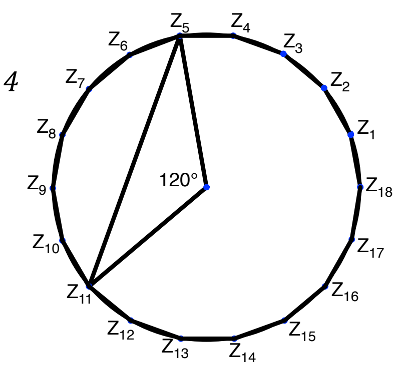
$\boxed{5 \le x \le 14}$ **or** $\boxed{[5,14]}$

3. Determine the set of all real numbers x such that $\sqrt{14-x} - \sqrt{x-1} \le 1$.

$\boxed{\sqrt{3}}$

4. $z^{18} = 1$ has 18 solutions in the complex plane of the form $r(\cos\theta + i\sin\theta)$. Let z_1 be the solution with the smallest positive angle, z_2 be the next largest angle, and so on. When graphed in the complex plane, the points z_5, z_7, and z_{11} are the vertices of a triangle. Calculate the length of the segment from z_5 to z_{11}.

Figure 4



Area = $\boxed{4\sqrt{35}}$

5. In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$, and the diagonals intersect at P. If $AB = AP = DP = \sqrt{CP} = 3$, determine exactly the area of $ABCD$.

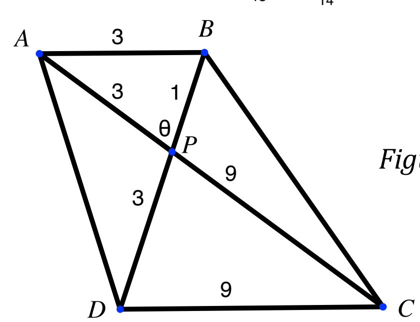


Figure 5

Area = $\boxed{44}$

6. What is the area bounded by $y = 3x$, $y = \frac{1}{4}x$, $y = 3x - 11$, and $y = \frac{1}{4}x + 11$?

Minnesota State High School Mathematics League

2016-17 Meet 3, Team Event

SOLUTIONS (page 2)

1. *The numbers sum to 20 so x + y + z = 20. Since the sum of the pairwise differences of the three numbers is 18 then (x – y) + (y – z) + (x – z) = 2x – 2z = 18. Simplifying our second equation and solving for z gives z = x – 9. Substituting for z gives us x + y + (x – 9) = 2x + y = 29. Since x > y and y > z, the only possible values for x and y are (12, 5), (11, 7), and (10, 9). The only pair where both numbers are not prime is (10, 9), so the ordered triple is* $\boxed{(10,9,1)}$.

2. *First apply the Law of Sines to* $\triangle CDE$ *we get* $\frac{CE}{\sin 120°} = \frac{\sqrt{6}}{\sin 45°} \Rightarrow CE = \frac{\sqrt{18}}{\sqrt{2}} = 3$. *Applying the Law of Cosines to* $\triangle ACE$ *gives* $AE^2 = 3^2 + \left(\sqrt{6}-\sqrt{2}\right)^2 - 2(3)\left(\sqrt{6}-\sqrt{2}\right)\cos 165°$. But $\cos 165° = \cos\left(120° + 45°\right) = \frac{-\sqrt{6}-\sqrt{2}}{4}$. *Therefore,*
$$AE^2 = 3^2 + \left(\sqrt{6}-\sqrt{2}\right)^2 - 2(3)\left(\sqrt{6}-\sqrt{2}\right)\cos 165° = 9 + 8 - 4\sqrt{3} + 6 = 23 - 4\sqrt{3} \Rightarrow AE = \sqrt{23-4\sqrt{3}} \approx \boxed{4.009}.$$

3. *Start by solving the equality part. This gives us* $\sqrt{14-x} - \sqrt{x-1} = 1 \Rightarrow \sqrt{14-x} = 1 + \sqrt{x-1}$. *Squaring and rearranging gives* $\sqrt{14-x} = 1 + \sqrt{x-1} \Rightarrow 14 - x = 1 + 2\sqrt{x-1} + x - 1 \Rightarrow 7 - x = \sqrt{x-1} \Rightarrow x^2 - 15x + 50 = 0 \Rightarrow (x-5)(x-10) = 0 \Rightarrow x = 5$ *or* $x = 10$. *As solutions to the equality, 5 will work but 10 does not. The domain of the inequality is* $1 \le x \le 14$. *As x moves from 5 towards 1,* $\sqrt{14-x}$ *gets larger while* $\sqrt{x-1}$ *is getting smaller. This means the difference will be greater than 1 and* $1 \le x < 5$ *is not part of the solution set. As x moves from 5 to 14,* $\sqrt{14-x}$ *is getting smaller and* $\sqrt{x-1}$ *is getting larger so the difference is smaller than 1 and will eventually become negative. Therefore, the solution set is* $\boxed{5 \le x \le 14}$.

4. *The solutions to* $z^{18} = 1$ *are all points on a unit circle of radius 1 in the complex plane, as shown in Figure 4. These points are separated by equal angle measurements of 20°. The triangle formed by* z_5, z_{11} *and the origin is isosceles with a 120° angle at the apex. Using the Law of Cosines,* $\left(z_5 z_{11}\right)^2 = 1^2 + 1^2 - 2(1)(1)\cos 120° = 3 \Rightarrow z_5 z_{11} = \boxed{\sqrt{3}}$.

5. *See Figure 5. Since* $\triangle PAB$ *and* $\triangle PCD$ *are similar we can deduce that their side lengths are 3, 3, 1 and 9, 9, 3, respectively. The area of a trapezoid with diagonals of lengths a and b and which meet at an angle of θ is .5absin θ. Using the Law of Cosines on* $\triangle PAB$ *we get* $3^2 = 3^2 + 1^2 - 2(3)(1)\cos\theta \Rightarrow \cos\theta = \frac{1}{6}$. *Applying the Pythagorean Identity gives* $\sin\theta = \frac{\sqrt{35}}{6}$. *Thus, the area of ABCD is .5 • (1 + 3) • (3 + 9) •* $\frac{\sqrt{35}}{6}$ = $\boxed{4\sqrt{35}}$.

6. *First find the points where these lines intersect. These points can be found either algebraically or graphically and are (0,0), (4,12), (4,1), and (8,13). Since the slopes of y = 3x – 11 and y = 3x are the same, these lines are parallel. The slopes of the other two lines are the same so those lines are parallel as well. The region bounded these lines is a parallelogram whose area is the length of its base multiplied by the height. We shall call the segment between (0,0) and (4,12) the base. The length of this segment is* $\sqrt{4^2 + 12^2} = \sqrt{160}$. *The height is the distance between the point (0,0) and the line y = 3x – 11. This distance is* $\frac{\left|3 \bullet 0 + -1 \bullet 0 - 11\right|}{3^2 + 1^2} = \frac{11}{\sqrt{10}}$. *Therefore, the area of the parallelogram is* $\sqrt{160} \bullet \frac{11}{\sqrt{10}} = \sqrt{\frac{160}{10}} \bullet 11 = \sqrt{16} \bullet 11 = \boxed{44}$.