Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

__________ *fleeps* 1. If two fleeps and three blorps are worth the same as six fleeps and one blorp, how many fleeps is a blorp worth?

__________ 2. How many integers n satisfy the inequalities $2n < 7n - 5 \leq 6n$?

__________ 3. Alec mowed lawns for his summer job each of the past 5 summers. Each summer after the first, he mowed 5 more lawns and charged $2 more per lawn than during the previous summer. Determine how much Alec charged per lawn his first summer if he mowed 85 lawns over the course of those summers and made a total of $1800.

__________ 4. Positive integers a and b are called *equalish* if $a \neq b$ and $|a-b| < \min\{a,b\}$. Find the number of integers that are *equalish* to 2016 but not *equalish* to 1026.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event A

SOLUTIONS

2 *fleeps* 1. If two fleeps and three blorps are worth the same as six fleeps and one blorp, how many fleeps is a blorp worth?

Let F = fleeps and B = blorps. Therefore, 2F + 3B = 6F + 1B ⇒ 2B = 4F ⇒ B = 2F.

4 2. How many integers n satisfy the inequalities $2n < 7n - 5 \le 6n$?

There are two inequalities present here. The first inequality is $2n < 7n - 5 \Rightarrow 5 < 5n \Rightarrow 1 < n$, *while the second inequality is* $7n - 5 \le 6n \Rightarrow n \le 5$. *There are 4 integers that satisfy both, namely {2, 3, 4, and 5}.*

$16 3. Alec mowed lawns for his summer job each of the past 5 summers. Each summer after the first, he mowed 5 more lawns and charged $2 more per lawn than during the previous summer. Determine how much Alec charged per lawn his first summer if he mowed 85 lawns over the course of those summers and made a total of $1800.

If r = the number of lawns Alec mows the first summer, then r + 5, r + 10, r + 15, and r + 20 are the number of lawns he mows the next four summers. This means r + (r + 5) + (r + 10) + (r + 15) + (r + 20) = 85 ⇒ 5r + 50 = 85 ⇒ r = 7. So, he mows 7, 12, 17, 22, and 27 lawns the five summers. Let d be the rate he charges for the first summer. Then his rates for the five summers are d, d + 2, d + 4, d + 6, and d + 8, and his total profit is 7d + 12(d + 2) + 17(d + 4) + 22(d + 6) + 27(d + 8) = 1800 ⇒ 85d + 440 = 1800 ⇒ d = 16. Therefore, Alec charges $16 per lawn his first summer.

1981 4. Positive integers a and b are called *equalish* if $a \ne b$ and $|a - b| < \min\{a, b\}$. Find the number of integers that are *equalish* to 2016 but not *equalish* to 1026.

The set {1, 2, 3, … ,1008} are not equalish to 2016 since $|a - 2016| < \min\{a, 2016\}$ *is false for each number in that set. The positive integers that are equalish to 2016 are {1009, 1010, … 2014, 2015, 2017, 2018, … ,4031}. The integer 2016 does not work since it will result in the false inequality 0 < 0. The positive integers that are equalish to 1026 are {514, 515, … ,1024, 1025, 1027, 1028, … ,2051}. Be careful here as well since 1026 will also give the false statement of 0 < 0. The set of positive integers equalish to 2016 but not 1026 are {1026, 2052, 2053, … ,4031}. There are 1981 integers in this set.*

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

_______________ 1. Determine exactly the area of a triangle whose sides have lengths 41, 9, and 40 .

_______________ 2. A pyramid has a rectangular base with a width that is one half the length. If the height of the pyramid is 4 cm and its volume is 24 cm^3, determine exactly the length of the base.

_______________ 3. Figure 3 shows $\triangle ABC$ with side lengths AB = 7, BC = 5, and AC = 3. Determine exactly the length of the altitude from C to $\overline{AB}$.

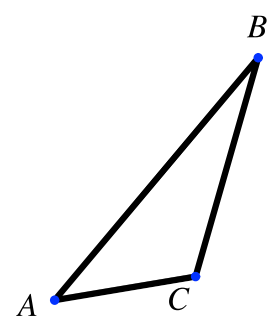
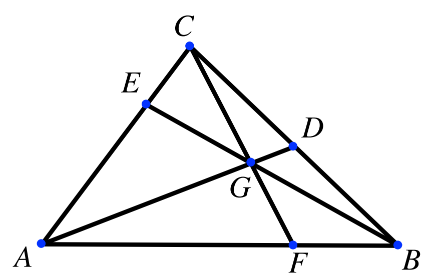


Figure 3

_______________ 4. In Figure 4 $\triangle ABC$ has side lengths AC= 9, BC = 8, and AB = 12. Let D lie on $\overline{BC}$ such that $\overline{AD}$ is a median. Segment $\overline{BE}$ is constructed with E between A and C and CE = 2. Let $\overline{AD}$ and $\overline{BE}$ intersect at G. If a line segment from C through G intersects $\overline{AB}$ at F, determine exactly the length of $\overline{AF}$.

Figure 4



Name: ____________________________ Team: ____________________________

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event B

SOLUTIONS

Answer: 180

1. Determine exactly the area of a triangle whose sides have lengths 41, 9, and 40 .

The sides form a Pythagorean triple so the area is $\frac{1}{2}bh=\frac{1}{2}\cdot 9\cdot 40=180.$

Answer: 6

2. A pyramid has a rectangular base with a width that is one half the length. If the height of the pyramid is 4 cm and its volume is 24 cm^3, determine exactly the length of the base.

The volume of a pyramid is $V=\frac{1}{3}\bullet \textit{base area}\bullet \textit{height}$. *Let x be the length of the rectangular base. Therefore, the area of the base is* $\frac{1}{2}x^2$ *and by substitution into volume equation* $24=\frac{1}{3}\bullet\frac{1}{2}x^2\bullet 4 \Rightarrow x^2=36 \Rightarrow x=6.$

Answer: $\frac{15\sqrt{3}}{14}$

3. Figure 3 shows $\triangle ABC$ with side lengths AB = 7, BC = 5, and AC = 3. Determine exactly the length of the altitude from C to $\overline{AB}$.

Using Heron's formula, the area of $\triangle ABC$ *is*

$$\sqrt{\frac{15}{2}\left(\frac{15}{2}-7\right)\left(\frac{15}{2}-5\right)\left(\frac{15}{2}-3\right)}=\frac{15\sqrt{3}}{4}=\frac{1}{2}\bullet AB\bullet h\Rightarrow\frac{15\sqrt{3}}{4}=\frac{1}{2}\bullet 7\bullet h\Rightarrow h=\frac{15\sqrt{3}}{14}.$$

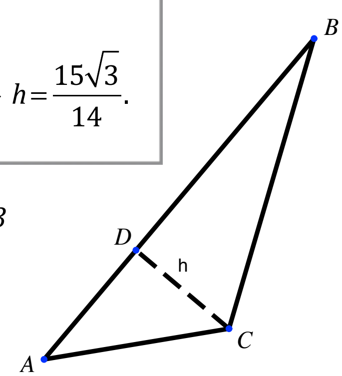


Figure 3

Answer: $\frac{28}{3}$ *or* $9\frac{1}{3}$ *or* $9.\overline{3}$

4. In Figure 4 $\triangle ABC$ has side lengths AC= 9, BC = 8, and AB = 12. Let D lie on $\overline{BC}$ such that $\overline{AD}$ is a median. Segment $\overline{BE}$ is constructed with E between A and C and CE = 2. Let $\overline{AD}$ and $\overline{BE}$ intersect at G. If a line segment from C through G intersects $\overline{AB}$ at F, determine exactly the length of $\overline{AF}$.

See Figure 4. Using Ceva's Theorem we have

$$\frac{7}{2}\bullet\frac{4}{4}\bullet\frac{12-AF}{AF}=1\Rightarrow 84-7\bullet AF=2\bullet AF\Rightarrow AF=\frac{28}{3}.$$

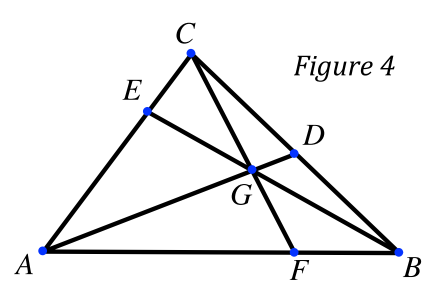


Figure 4

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________ 1. Determine the least positive radian value of x such that $\cos 4x \cos x + \sin 4x \sin x = \cos\frac{3\pi}{5}$.

$\sec 2\theta =$ ________ 2. If $\pi < \theta < \frac{3\pi}{2}$ and $\sin\theta = -\frac{8}{17}$, determine exactly the value of $\sec 2\theta$.

________ 3. In $\triangle ABC$, if $\tan A = 2$ and $\tan B = 3$, determine tan C exactly.

cos x = ________ 4. If x is acute and $\tan\frac{x}{2} + \cot\frac{x}{2} = 5\cot x$, determine exactly the value of $\cos x$.

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\frac{\pi}{5}}$

1. Determine the least positive radian value of x such that $\cos 4x\cos x+\sin 4x\sin x=\cos\frac{3\pi}{5}$.

This is the difference identity for cosine:

$$\cos 4x\cos x+\sin 4x\sin x=\cos\left(4x-x\right)=\cos 3x=\cos\frac{3\pi}{5}\Rightarrow x=\frac{\pi}{5}.$$

$\sec 2\theta=\boxed{\frac{289}{161}}$ ***or*** $\boxed{1\frac{128}{289}}$

2. If $\pi<\theta<\frac{3\pi}{2}$ and $\sin\theta=-\frac{8}{17}$, determine exactly the value of $\sec 2\theta$.

$$\sec 2\theta=\frac{1}{\cos 2\theta}=\frac{1}{1-2\sin^2\theta}=\frac{1}{1-2\left(\frac{-8}{17}\right)^2}=\frac{1}{\left(\frac{161}{289}\right)}=\frac{289}{161}.$$

tan C = $\boxed{1}$

3. In $\triangle ABC$, if tan A = 2 and tan B = 3, determine tan C exactly.

Since this is a triangle A + B + C = 180° ⇒ C = 180° –(A + B). Using the difference identity for tangent, we get $\tan C=\tan\left(180°-\left(A+B\right)\right)=-\tan\left(A+B\right)$. *Applying the addition identity for tangent gives* $-\tan\left(A+B\right)=-\frac{\tan A+\tan B}{1-\tan A\tan B}=-\frac{2+3}{1-2\bullet 3}=1.$

cos x = $\boxed{\frac{2}{5}}$

4. If x is acute and $\tan\frac{x}{2}+\cot\frac{x}{2}=5\cot x$, determine exactly the value of $\cos x$.

Rewriting the equation in terms of sine and cosine we obtain $\tan\frac{x}{2}+\cot\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\cos\frac{x}{2}\bullet\sin\frac{x}{2}}.$

Applying the Pythagorean identity and the double angle identity for sine on the numerator and denominator respectively we get $\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\cos\frac{x}{2}\bullet\sin\frac{x}{2}}=\frac{1}{\frac{1}{2}\bullet 2\cos\frac{x}{2}\bullet\sin\frac{x}{2}}=\frac{2}{\sin x}$. *So,* $\frac{2}{\sin x}=\frac{5\cos x}{\sin x}\Rightarrow\cos x=\frac{2}{5}.$

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

_____________ 1. What is the distance between the intercepts of the line $3x + 4y = 12$?

_____________ 2. Determine exactly the slope of the line that passes through the origin and is perpendicular to $2x - 5y = 10$.

_____________ 3. Determine exactly the area between the x-axis, $y = \frac{1}{3}x + 4$, and $y = -2x + 7$.

_____________ 4. Perpendicular lines ℓ_1 and ℓ_2 intersect at point T on the line $y = x$. If ℓ_1 has a y-intercept of (0,3) and ℓ_2 has an x-intercept of (7,0), determine exactly the x-coordinate of *T*.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2016-17 Meet 2, Individual Event D

SOLUTIONS

$\boxed{5}$ 1. What is the distance between the intercepts of the line $3x + 4y = 12$?

The intercepts of the line $3x + 4y = 12$ are (4,0) and (0,3). So, the distance between these points is $\sqrt{3^2+4^2}=5$.

$\boxed{\frac{-5}{2}}$ **or** $\boxed{-2\frac{1}{2}}$ **or** $\boxed{-2.5}$ 2. Determine exactly the slope of the line that passes through the origin and is perpendicular to $2x - 5y = 10$.

The line $2x - 5y = 10$ has a slope of $\frac{2}{5}$. The slope of the line perpendicular is then $\frac{-5}{2}$.

$\boxed{\frac{961}{28}}$ **or** $\boxed{34\frac{9}{28}}$ 3. Determine exactly the area between the x-axis, $y=\frac{1}{3}x+4$, and $y = -2x + 7$.

Setting $y = 0$ in each of the above equations gives x-intercepts of (−12,0) and $(\frac{7}{2},0)$ respectively. These form the base of our triangle as shown in figure 3.
The point of intersection of these lines, which gives the height of the triangle, occurs when $-2x+7=\frac{1}{3}x+4 \Rightarrow x=\frac{9}{7}$. So, $y = \frac{31}{7}$ and the area of the triangle is $\frac{1}{2}\bullet\left(12+\frac{7}{2}\right)\bullet\frac{31}{7}=\frac{961}{28}$.

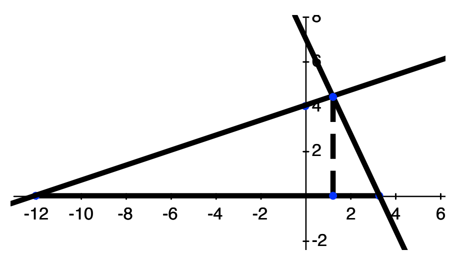


Figure 3

$x = \boxed{5}$ 4. Perpendicular lines ℓ_1 and ℓ_2 intersect at point T on the line $y = x$. If ℓ_1 has a y-intercept of (0,3) and ℓ_2 has an x-intercept of (7,0), determine exactly the x-coordinate of *T*.

The point of intersection T is of the form (x,x) since the point is on the line y = x. The slopes of ℓ_1 and ℓ_2 *are* $\frac{x-3}{x}$ *and* $\frac{x}{x-7}$ *respectively. Equating one to the negative reciprocal of the other gives* $\frac{x-3}{x}=-\frac{x-7}{x} \Rightarrow \frac{x-3}{x}=\frac{7-x}{x} \Rightarrow 2x=10 \Rightarrow x=5$.

Minnesota State High School Mathematics League

2016-17 Meet 2, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

____________ 1. Lines ℓ_1 and ℓ_2 pass through (0, 10) and (0, –4) respectively. If the lines are perpendicular and have the same positive x-intercept, determine exactly the x-intercept.

____________ 2. Positive integers a, b, c, and d satisfy $a < b < c < d < 2016$ and $2016 < 2d < 3c < 4b < 5a$. What is the largest possible value for $a + b + c - d$?

____________ 3. In *Figure 3* rectangle $ABCD$, $AD = 6$, $AB = 8$, $\overline{DP}$ bisects $\angle BDC$ and $\overline{AT}$ is perpendicular to $\overline{DP}$ at T. Determine exactly the ratio of the length of $\overline{AT}$ to the length of $\overline{DT}$.

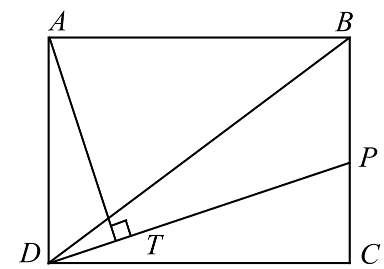


Figure 3

____________ 4. A pyramid has a triangular base ABC. The apex D is directly above and at a distance of 8 from the centroid O of $\triangle ABC$ and $m\angle DOA = 90°$. If E is the midpoint of $\overline{AB}$, $AO = 9$, $AB = 14$, and $EO = 4$, determine exactly the volume of the pyramid.

____________ 5. Let $0° < A < 90°$ and $90° < B < 180°$. If both angles are positive integers and $\cos B(\cos B - \cos 3A) = \sin B(\sin B + \sin 3A)$, determine the number of ordered pairs (A, B).

$x + y =$ ____________ 6. Determine exactly the sum of the x and y values of the point on $3x + 4y = 12$ that is closest to the origin.

Team: ________________________

Minnesota State High School Mathematics League

2016-17 Meet 2, Team Event

SOLUTIONS (page 1)

$\boxed{\left(2\sqrt{10},0\right)}$

Graders: Also accept $2\sqrt{10}$

1. Lines ℓ_1 and ℓ_2 pass through (0, 10) and (0, –4) respectively. If the lines are perpendicular and have the same positive x-intercept, determine exactly the x-intercept.

$\boxed{4024}$

2. Positive integers a, b, c, and d satisfy $a < b < c < d < 2016$ and $2016 < 2d < 3c < 4b < 5a$. What is the largest possible value for $a + b + c - d$?

$\boxed{3}$

3. In *Figure 3* rectangle $ABCD$, $AD = 6$, $AB = 8$, $\overline{DP}$ bisects $\angle BDC$ and $\overline{AT}$ is perpendicular to $\overline{DP}$ at T. Determine exactly the ratio of the length of $\overline{AT}$ to the length of $\overline{DT}$.

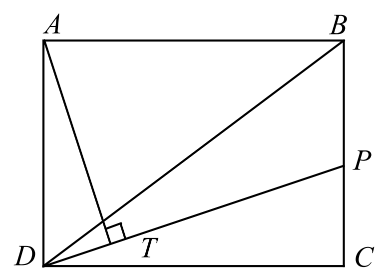


Figure 3

$\boxed{96\sqrt{5}}$

4. A pyramid has a triangular base ABC. The apex D is directly above and at a distance of 8 from the centroid O of $\triangle ABC$ and $m\angle DOA = 90°$. If E is the midpoint of $\overline{AB}$, $AO = 9$, $AB = 14$, and $EO = 4$, determine exactly the volume of the pyramid.

$\boxed{88}$

5. Let $0° < A < 90°$ and $90° < B < 180°$. If both angles are positive integers and $\cos B\left(\cos B - \cos 3A\right) = \sin B\left(\sin B + \sin 3A\right)$, determine the number of ordered pairs (A,B).

$x + y = \boxed{\frac{84}{25}}$ *or* $\boxed{3\frac{9}{25}}$ *or* $\boxed{3.36}$

6. Determine exactly the sum of the x and y values of the point on 3x + 4y = 12 that is closest to the origin.

Minnesota State High School Mathematics League

2016-17 Meet 2, Team Event

SOLUTIONS (page 2)

1. *Let (a, 0) be the common x-intercept. The slope of the line through (0, 10) and (a, 0) is* $\frac{-10}{a}$. *The slope of the line through (0, –4) and (a, 0) is* $\frac{4}{a}$. *Since the lines are perpendicular, their slopes are opposite reciprocals of one another. Equating one slope to the opposite reciprocal of the other gives* $\frac{a}{10}=\frac{4}{a} \Rightarrow a^2=40 \Rightarrow a=\sqrt{40}=\boxed{2\sqrt{10}}$.

2. *The aim is to find the largest possible value for a + b + c – d = a + b + (c – d). The difference, c – d, will be negative since c < d. We want this difference to be very small, preferably –1. We also want to make a + b as large as possible. By our second chain of inequalities 2016 < 2d ⟹ d > 1008. By our first chain of inequalities d < 2016. Choosing a d value closer to 1008 will make a and b and thus a + b smaller than if we choose something closer to 2016. If d is 2015, then a c value of 2014 would make our c – d difference –1. To maximize our sum of a + b, a would have to be 2012 and b would have to be 2013. Does this satisfy our second chain of inequalities? It does, so a + b + c – d = 2012 + 2013 + 2014 – 2015 =* $\boxed{4024}$.

3. *Let* $m\angle BDC=\alpha$ *and* $m\angle ADB=\beta$. *By the Angle Bisector Theorem, we have* $\frac{PC}{PB}=\frac{DC}{DB} \Rightarrow \frac{PC}{6-PC}=\frac{8}{10} \Rightarrow PC=\frac{8}{3}$. *Then* $\tan\angle PDC=\tan\frac{\alpha}{2}=\frac{\frac{8}{3}}{8}=\frac{1}{3}=\tan\angle BDP$. *Since* $\triangle ADB$ *is a right triangle,* $\tan\beta=\frac{8}{6}=\frac{4}{3}$. *Therefore, the desired ratio* $\frac{AT}{DT}$ *is* $\tan\left(\frac{\alpha}{2}+\beta\right)$. *Using the tangent addition identity and substitution gives* $\tan\left(\frac{\alpha}{2}+\beta\right)=\frac{\tan\frac{\alpha}{2}+\tan\beta}{1-\tan\frac{\alpha}{2}\tan\beta}=\frac{\frac{1}{3}+\frac{4}{3}}{1-\frac{1}{3}\bullet\frac{4}{3}}=\boxed{3}$.

4. *The centroid of a triangle is the intersection of the medians which split* $\triangle ABC$ *into 6 smaller triangles of equal area. Using Heron's formula on* $\triangle AOE$ *with AO = 9, OE = 4, and AE =* $\frac{1}{2}\bullet 14=7$, *we get an area of* $\sqrt{10(10-9)(10-7)(10-4)}=6\sqrt{5}$. *The volume of the pyramid is* $\frac{1}{3}\bullet\left(6\bullet 6\sqrt{5}\right)\bullet 8=\boxed{96\sqrt{5}}$.

5. *Distributing we get* $\cos B(\cos B-\cos 3A)=\sin B(\sin B+\sin 3A) \Rightarrow \cos^2 B-\cos B\cos 3A=\sin^2 B+\sin B\sin 3A$. *Rewriting and applying both the double angle identity for sine and the difference identity for cosine gives us* $\cos^2 B-\cos B\cos 3A=\sin^2 B+\sin B\sin 3A \Rightarrow \cos^2 B-\sin^2 B=\cos B\cos 3A+\sin B\sin 3A \Rightarrow \cos 2B=\cos(3A-B)$. *There are two possibilities here. The first possibility is that* $2B+360n^\circ=3A-B \Rightarrow 3B=3A+360n^\circ \Rightarrow B=A+120n^\circ$, *where n is an integer. Any n larger than 1 is not possible since it will make B too high. This means* $B=A+120^\circ$, *so* $1^\circ \le A \le 59^\circ$ *for 59 possible A values. The second possibility is that* $2B+(3A-B)=360n^\circ$, *which means* $B+3A=360n^\circ \Rightarrow B=360n^\circ-3A$. *Again, n must be 1 since any value larger 1 will make B too large. So* $B=360^\circ-3A$. *Since B is obtuse, then* $90^\circ < B < 180^\circ$, *so A must be larger than* 60° *but smaller than* 90°. *Therefore, there are 29 + 59 such A values for* $\boxed{88}$ *ordered pairs (A,B).*

6. *The point closest to the origin will lie on a line through the origin that is perpendicular to 3x + 4y = 12. Since 3x + 4y =12 has a slope of* $\frac{-3}{4}$, *the slope of the perpendicular line will be* $\frac{4}{3}$. *The equation the perpendicular line is* $y=\frac{4}{3}x$. *Using substitution of the second equation into the first gives* $3x+4\left(\frac{4}{3}x\right)=12 \Rightarrow 3x+\frac{16x}{3}=12 \Rightarrow x=\frac{36}{25}$. *Substituting again we get* $y=\frac{48}{25}$, *so* $x+y=\boxed{\frac{84}{25}}$.