Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Express $\frac{2}{3}+\frac{5}{\frac{5}{3}+\frac{5}{6}}$ as the quotient of two relatively prime integers.

________________ 2. Find the base-nine number that is equivalent to 245_6.

min $x =$ __________ 3. If 48 and x have a lowest common multiple of 2640 and a greatest common factor of 12, determine the minimum possible value of x.

$\underline{A}\ \underline{B}\ \underline{C} =$ __________ 4. Let $\underline{A}\ \underline{B}\ \underline{C}$ be a three-digit number where $\underline{A}$, $\underline{B}$, and $\underline{C}$ are distinct and $\underline{A} + \underline{B} + \underline{C} = 20$. If $\underline{A}\ \underline{B}\ \underline{C}$ is split into a two-digit number $\underline{A}\ \underline{B}$ and the one-digit number $\underline{C}$, then $\underline{C}$ is the greatest common factor of $\underline{A}\ \underline{B}$ and $\underline{C}$, and $\underline{A}\ \underline{B}$ is the least common multiple. What is $\underline{A}\ \underline{B}\ \underline{C}$?

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\frac{8}{3}}$

Graders: ***<u>No</u> alternate forms accepted here.***

1. Express $\frac{2}{3}+\frac{5}{\frac{5}{3}+\frac{5}{6}}$ as the quotient of two relatively prime integers.

$$\frac{6}{6}\left(\frac{2}{3}+\frac{5}{\frac{5}{3}+\frac{5}{6}}\right)=\frac{12}{18}+\frac{30}{10+5}=\frac{12}{18}+2=\frac{12}{18}+\frac{36}{18}=\frac{48}{18}=\frac{8}{3}.$$

$\boxed{122}$

(Student does <u>not</u> necessarily need to include the subscript "9" to indicate the base.)

2. Find the base-nine number that is equivalent to 245_6.

$$2\bullet 6^2+4\bullet 6^1+5\bullet 6^0=101 \textit{ and } 101=1\bullet 9^2+2\bullet 9^1+2\bullet 9^0=122_9$$

$\min x = \boxed{660}$

3. If 48 and *x* have a lowest common multiple of 2640 and a greatest common factor of 12, determine the minimum possible value of *x*.

$48=2^4\bullet 3, 12=2^2\bullet 3$ *and* $2640=2^4\bullet 3\bullet 5\bullet 11$. *The GCF(48, x) = 12, so x is a multiple of 12. Since LCM(48, x) = 2640, x must have at least one factor of 5 and one factor of 11. Therefore, the minimum possible value for x is* $2^2\bullet 3\bullet 5\bullet 11=660$.

$\underline{A}\,\underline{B}\,\underline{C} = \boxed{497}$

4. Let $\underline{A}\,\underline{B}\,\underline{C}$ be a three-digit number where $\underline{A}$, $\underline{B}$, and $\underline{C}$ are distinct and $\underline{A}+\underline{B}+\underline{C}=20$. If $\underline{A}\,\underline{B}\,\underline{C}$ is split into a two-digit number $\underline{A}\,\underline{B}$ and the one-digit number $\underline{C}$, then $\underline{C}$ is the greatest common factor of $\underline{A}\,\underline{B}$ and $\underline{C}$, and $\underline{A}\,\underline{B}$ is the least common multiple. What is $\underline{A}\,\underline{B}\,\underline{C}$?

C can not be 0, 1, or 2 since $\underline{A}$, $\underline{B}$, and $\underline{C}$ are distinct and $\underline{A}+\underline{B}+\underline{C}=20$. A chart for the possible C values is shown below:

C	**3**	**4**	**5**	**6**	**7**	**8**	**9**
(A,B)	(8,9)	(7,9) (9,7)	(6,9) , (7,8) (9,6) , (8,7)	(5,9) (9,5)	(4,9) , (9,4) (5,8) , (8,5)	(3,9) , (9,3) (5,7) , (7,5)	(3,8) , (8,3) , (4,7) (7,4) , (5,6) , (6,5)
Is $\underline{C}$ the GCF of $\underline{A}\,\underline{B}$?	No	No	No	No	Yes, for 49	No	No
Is $\underline{A}\,\underline{B}$ the LCM?	No	No	No	No	Yes, for 49	No	No

Therefore, $\underline{A}\,\underline{B}\,\underline{C}$ is 497.

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

__________ 1. A rectangular box has faces whose side lengths are $\sqrt{2}$, 3, and 5. Find the longest diagonal of the box.

$AD =$ __________ 2. $\triangle ABC$ is an isosceles right triangle whose hypotenuse $\overline{AC}$ has a length of $9\sqrt{6}$. If point D lies on $\overline{BC}$ such that $m\angle BAD = 30°$, determine exactly AD.

__________ 3. Given $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = y$, as shown in *Figure 3*. Find the <u>smallest</u> possible angle y *(in degrees)* if x is an obtuse angle with an integer measure.

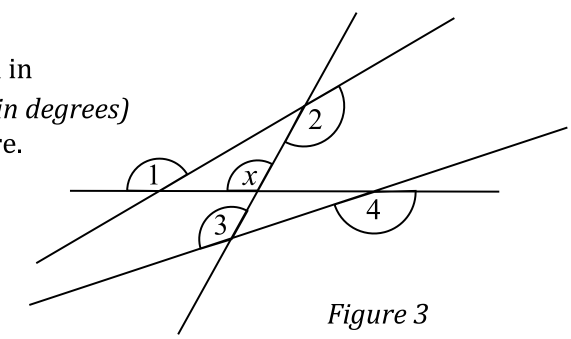

Figure 3

__________ 4. Rectangle $ABCD$ has sides of length 6 and 8. Exterior equilateral triangles ABP, BCM, CDN, and ADO are formed, as shown in *Figure 4*. Calculate the perimeter of quadrilateral $MNOP$.

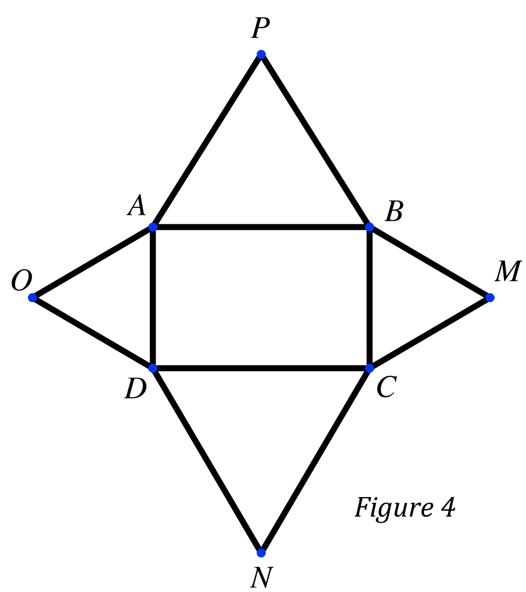

Figure 4

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event B

SOLUTIONS

Answer: 6

1. A rectangular box has faces whose side lengths are $\sqrt{2}$, 3, and 5. Find the longest diagonal of the box.

> *The length of the longest diagonal of a box is* $\sqrt{l^2+w^2+h^2}=\sqrt{\left(\sqrt{2}\right)^2+3^2+5^2}=\sqrt{36}=6.$

Answer: $AD = 18$

2. $\triangle ABC$ is an isosceles right triangle whose hypotenuse $\overline{AC}$ has a length of $9\sqrt{6}$. If point D lies on $\overline{BC}$ such that $m\angle BAD = 30°$, determine exactly AD.

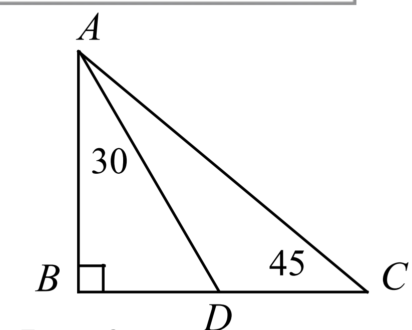

Figure 2

> *Since* $\triangle ABC$ *is isosceles and* $AC=9\sqrt{6}$, $AB=BC=\dfrac{9\sqrt{6}}{\sqrt{2}}=9\sqrt{3}$. *Triangle ADB is a 30°-60°-90° right triangle, so* $BD=\dfrac{9\sqrt{3}}{\sqrt{3}}=9$, *making AD = 18.*

Answer: 542°

3. Given $m\angle 1+m\angle 2+m\angle 3+m\angle 4=y$, as shown in *Figure 3*. Find the <u>smallest</u> possible angle *y (in degrees)* if *x* is an obtuse angle with an integer measure.

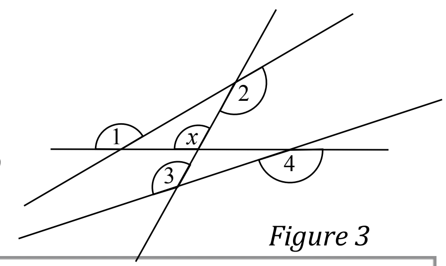

Figure 3

> *Exterior* $\angle 2$ *of the triangle containing x is the sum of its two non-adjacent interior angles. Therefore,* $m\angle 2=x+\left(180°-m\angle 1\right)\Rightarrow m\angle 1+m\angle 2=x+180°$. *Using similar reasoning,* $m\angle 3+m\angle 4=x+180°$. *This means* $m\angle 1+m\angle 2+m\angle 3+m\angle 4=2x+360°=y\Rightarrow x=\dfrac{y}{2}-180°$. *The smallest angle y will be obtained when x is 91°. Thus, y = 2(91°+ 180°) = 542°.*

Answer: 54.131

4. Rectangle *ABCD* has sides of length 6 and 8. Exterior equilateral triangles *ABP*, *BCM*, *CDN*, and *ADO* are formed, as shown in *Figure 4*. Calculate the perimeter of quadrilateral *MNOP*.

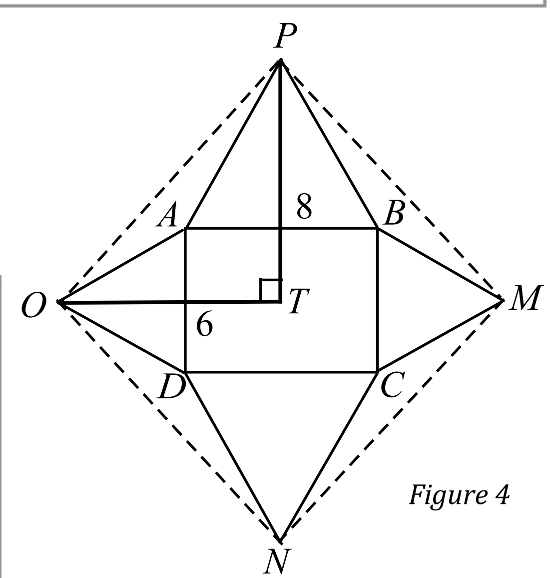

Figure 4

> *Drop altitudes from P and O and extend them to meet in T, as shown in Figure 4. Since triangle PAB is equilateral, its altitude is* $4\sqrt{3}$, *so* $PT=4\sqrt{3}+3$. *Triangle OAD is equilateral with an altitude of* $3\sqrt{3}$, *so* $OT=3\sqrt{3}+4$. *By the Pythagorean Theorem,*
>
> $OP=\sqrt{\left(4\sqrt{3}+3\right)^2+\left(3\sqrt{3}+4\right)^2}=\sqrt{100+48\sqrt{3}}=2\sqrt{25+12\sqrt{3}}.$
>
> *Therefore, the perimeter of MNOP is* $8\sqrt{25+12\sqrt{3}}\approx 54.131$.

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Determine exactly the value of $\sin\theta+\cos\theta$ if $\theta=\frac{5\pi}{4}$.

$\cos x=$ ____________ 2. If $\sin x=\frac{1}{3}$ and $0<x<\frac{\pi}{2}$, determine exactly the value of $\cos x$.

$\tan A=$ ____________ 3. If $\sin^2 A=\frac{9}{16}$ and A is in the second quadrant, determine exactly the value of $\tan A$.

________________ 4. In *Figure* 4 square *ABCD* is inscribed in square *PQRS*. If $\sin\angle BQC=\frac{1}{3}$, determine exactly $\tan\angle ADP$.

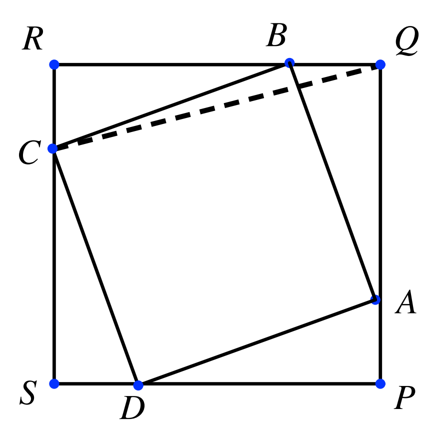

Figure 4

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$-\sqrt{2}$

1. Determine exactly the value of $\sin\theta+\cos\theta$ if $\theta=\frac{5\pi}{4}$.

$\sin\frac{5\pi}{4}+\cos\frac{5\pi}{4}=\frac{-\sqrt{2}}{2}+\frac{-\sqrt{2}}{2}=-\sqrt{2}$.

$\cos x = \frac{2\sqrt{2}}{3}$

2. If $\sin x=\frac{1}{3}$ and $0<x<\frac{\pi}{2}$, determine exactly the value of $\cos x$.

By the Pythagorean Identity:

$1=\left(\frac{1}{3}\right)^2+\cos^2 x \Rightarrow \cos x=\pm\sqrt{1-\frac{1}{9}}=\pm\frac{2\sqrt{2}}{3}$. *Since x is an acute angle, x =* $\frac{2\sqrt{2}}{3}$.

$\frac{-3\sqrt{7}}{7}$

3. If $\sin^2 A=\frac{9}{16}$ and A is in the second quadrant, determine exactly the value of $\tan A$.

Since A is an angle in the second quadrant, $\sin A=\frac{3}{4}$. *See Figure 3, where a reference triangle has been created using a circle of radius 4. By the Pythagorean Theorem, the horizontal leg has length* $\sqrt{4^2-3^2}=\sqrt{16-9}=\sqrt{7}$, *and because it extends in the negative x direction, we assign it the value* $-\sqrt{7}$. *Thus*

$\tan A=\frac{opp}{adj}=\frac{3}{-\sqrt{7}}=\frac{-3\sqrt{7}}{7}$.

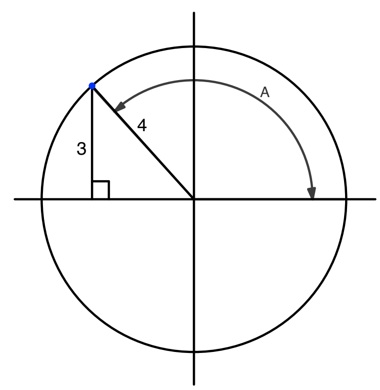

Figure 3

$\frac{2\sqrt{2}+1}{7}$

4. In *Figure* 4 square *ABCD* is inscribed in square *PQRS*. If $\sin\angle BQC=\frac{1}{3}$, determine exactly $\tan\angle ADP$.

Let BQ = x and BR = y. The other three sides of PQRS are split into segments of length x and y as well since all of the interior triangles are congruent to each other. Using the Pythagorean Theorem, $QC=\sqrt{(x+y)^2+x^2}$. *Thus*

$\frac{1}{3}=\sin\angle BQC=\frac{x}{\sqrt{(x+y)^2+x^2}}=\frac{1}{\sqrt{\left(1+\frac{y}{x}\right)^2+1}}\Rightarrow\sqrt{\left(1+\frac{y}{x}\right)^2+1}=3\Rightarrow\left(1+\frac{y}{x}\right)^2+1=9\Rightarrow\frac{y}{x}=2\sqrt{2}-1.$

Therefore, $\tan\angle ADP=\frac{x}{y}=\frac{1}{2\sqrt{2}-1}=\frac{2\sqrt{2}+1}{7}$.

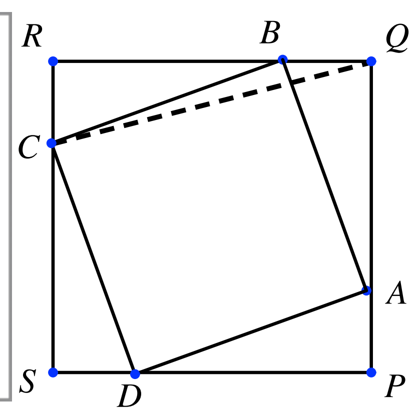

Figure 4

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

$x =$ ______________ 1. Determine exactly the product of the zeros of the equation $(2x-7)^2 = 36$.

$a =$ ______________ 2. For what value of a does the polynomial $3x^2 + ax + 10$ have 2 as a root?

$k =$ ______________ 3. Determine exactly all values of k for which the polynomials $x^2 + 2x - 5k$ and $x^2 - 10x - k$ share a common zero.

______________ 4. The function *f(x)* is a non-horizontal straight line where $f(x^2) = f(f(x))$ has two positive integer solutions at x = 2 and x = 5. Determine exactly *f(17)*.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2016-17 Meet 1, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\frac{13}{4}}$ *or* $\boxed{3\frac{1}{4}}$ *or* $\boxed{3.25}$

1. Determine exactly the product of the zeros of the equation $(2x-7)^2 = 36$.

Expand and rewrite to get $(2x-7)^2 = 36 \Rightarrow 4x^2 - 28x + 49 = 36 \Rightarrow 4x^2 - 28x + 13 = 0$. *The product of the roots is the constant term, divided by the leading coefficient:* $\frac{13}{4}$.

$\boxed{-11}$

2. For what value of *a* does the polynomial $3x^2 + ax + 10$ have 2 as a root?

A number is a root of a polynomial when substitution of that number into the polynomial yields a value of 0. Therefore, $3(2)^2 + a(2) + 10 = 0 \Rightarrow 22 + 2a = 0 \Rightarrow a = -11$.

$k = \boxed{0}$ *or* $k = \boxed{39}$

3. Determine exactly all values of k for which the polynomials $x^2 + 2x - 5k$ and $x^2 - 10x - k$ share a common zero.

Graders:
Award only 1 point if any <u>extra</u> solutions are given, or for only <u>one</u> correct solution.

If two functions share a common zero, so must their difference. This means $(x^2 + 2x - 5k) - (x^2 - 10x - k) = 12x - 4k$ *must be 0, so* $x = \frac{k}{3}$. *Substituting that value into the first equation gives* $\frac{k^2}{9} + \frac{2k}{3} - 5k = 0 \Rightarrow k^2 - 39k = 0 \Rightarrow k(k-39) = 0 \Rightarrow k = 0$ *or* $k = 39$.

$\boxed{109}$

4. The function *f(x)* is a non-horizontal straight line where $f(x^2) = f(f(x))$ has two positive integer solutions at x = 2 and x = 5. Determine exactly *f(17)*.

Let $f(x) = ax + b$. *Therefore,* $f(x^2) = ax^2 + b = f(f(x)) = a(ax + b) + b = a^2x + ab + b \Rightarrow b = x^2 - ax$.
Substituting x = 2 and x = 5 in, we get b = 4 – 2a and b = 25 – 5a. Equating these two equations gives a = 7 and b = –10. So, f(17) = 7(17) – 10 = 109.

Minnesota State High School Mathematics League

2016-17 Meet 1, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

$r =$ ______

$s =$ ______

1. The roots of x^2+bx+c are integers r and s, where $r < s < 0$. The greatest common factor of $|r|, |s|$, and c is 6 and the lowest common multiple of $|s|$ and b is 84. Find r and s.

______

2. In *Figure 2*, right triangle ABC has side lengths $AB = 6$, $BC = 12$. If the areas of EBD, AED, and ADC are equal, determine exactly the length of the altitude drawn from E to $\overline{AD}$.

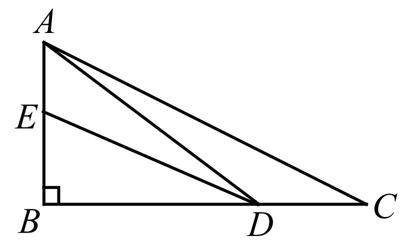

Figure 2

$x =$ ______

3. Find the smallest integer x such that $\sin(124_x) = \sin(221_x)$ where both angles are measured in degrees in base x.

______

4. A regular octagon $MNOPQRST$ has sides of length 6, as shown in *Figure 4*. Determine exactly $MQ^2 - MO^2$.

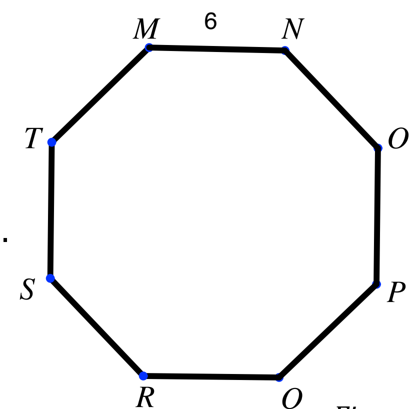

Figure 4

$a=$ ______

5. Determine the smallest positive integer $a > 2017$ for which $a^2 - 1$ can be expressed as the product of four consecutive integers.

$n=$ ______

6. If the sum of the solutions to $\sin(nx) = \frac{1}{2}$ on the interval $0 \le x < 2\pi$ is 139π, find n.

Team: ______________________

Minnesota State High School Mathematics League

2016-17 Meet 1, Team Event

SOLUTIONS (page 1)

$r = \boxed{-30}$

$s = \boxed{-12}$

1. The roots of $x^2 + bx + c$ are integers r and s, where $r < s < 0$. The greatest common factor of $|r|, |s|$, and c is 6 and the lowest common multiple of $|s|$ and b is 84. Find r and s.

$\boxed{\frac{12}{5}}$ *or* $\boxed{2\frac{2}{5}}$ *or* $\boxed{2.4}$

2. In *Figure 2*, right triangle ABC has side lengths $AB = 6$, $BC = 12$. If the areas of EBD, AED, and ADC are equal, determine exactly the length of the altitude drawn from E to $\overline{AD}$.

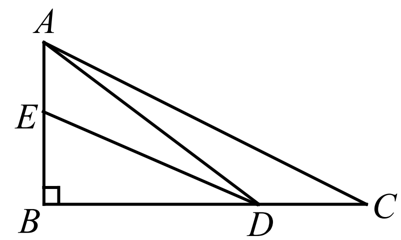

Figure 2

$x = \boxed{7}$

3. Find the <u>smallest</u> integer x such that $\sin(124_x) = \sin(221_x)$ where both angles are measured in degrees in base x.

$\boxed{72+36\sqrt{2}}$ *or* $\boxed{36(2+\sqrt{2})}$

4. A regular octagon $MNOPQRST$ has sides of length 6, as shown in *Figure 4*. Determine exactly $MQ^2 - MO^2$.

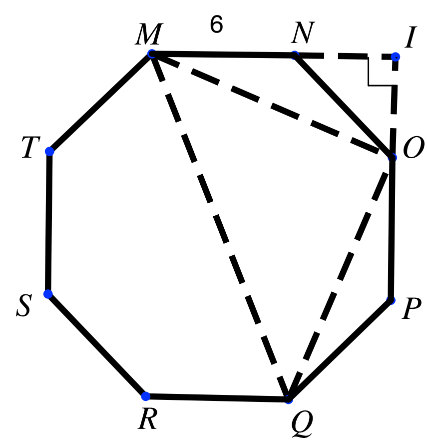

Figure 4

$a = \boxed{2069}$

5. Determine the smallest positive integer $a > 2017$ for which $a^2 - 1$ can be expressed as the product of four consecutive integers.

$\boxed{70}$

6. If the sum of the solutions to $\sin(nx) = \frac{1}{2}$ on the interval $0 \le x < 2\pi$ is 139π, find n.

Minnesota State High School Mathematics League

2016-17 Meet 1, Team Event

SOLUTIONS (page 2)

1. *If the greatest common factor of* $|r|, |s|$, *and c is 6, then r and s must be multiples of 6. Suppose* $|s| = 6$, *then b =28 or 84 since* $84 = 2^2 \bullet 3 \bullet 7$. *Since b = –(r + s),* $|r| = 22$ *or* 76. *However, neither of these are possible since the greatest common factor of* $|r|, |s|$, *and c would not be 6. If* $|s| = 12$, *then b = 28 or 42 or 84. This in turn means* $|r| = 16$ *or* 30 *or* 72. *The only value that works is 30 since the greatest common factor of 6 is required. Other possible values of* $|s|$ *like 42 and 84 are not possible since r < s < 0. Therefore, r =* $\boxed{-30}$ *and s =* $\boxed{-12}$.

2. *The area of ABC is* $\frac{1}{2} \bullet 6 \bullet 12 = 36$ *so the area of ADC is 12. Since the height of ADC is 6,* $\frac{1}{2} \bullet 6 \bullet DC = 12 \Rightarrow DC = 4$. *This means BD = 12 – 4 = 8. Using the Pythagorean Theorem,* $AD = \sqrt{6^2 + 8^2} = 10$. *If h is the altitude drawn from E to AD, then* $\frac{1}{2} \bullet 10 \bullet h = 12 \Rightarrow h = \boxed{\frac{12}{5}}$.

3. *The sine values of two angles are equal when those angles are either separated by a multiple of 360° or add to 180°. Assume the angles add to 180°. Thus,* $180 = x^2 + 2x + 4 + 2x^2 + 2x + 1 \Rightarrow (3x + 25)(x - 7) = 0 \Rightarrow x = \frac{-25}{3}$ *or* 7. *The only value that makes sense is* $\boxed{x = 7}$. *Note: If the angles are separated by a multiple of 360°, then* $360k + x^2 + 2x + 4 = 2x^2 + 2x + 1 \Rightarrow 360k + 3 = x^2$. *This gives x-values that are either non-integer or bigger than 7.*

4. *Extend sides* $\overline{MN}$ *and* $\overline{PO}$ *to meet a point I, as shown in Figure 4. This creates right triangle MIO. Let the sides of the octagon have length x. Since the measure of an interior of an octagon is 135°,* $m\angle ION$ *and* $m\angle INO$ *are 45° angles, making* $NI = OI = \frac{x}{\sqrt{2}}$. *By the Pythagorean Theorem,* $MO^2 = MI^2 + IO^2 = \left(x + \frac{x}{\sqrt{2}}\right)^2 + \left(\frac{x}{\sqrt{2}}\right)^2 = 2x^2 + x^2\sqrt{2}$. *The measure of* $m\angle OQP = 22.5°$ *and* $m\angle MQP = 67.5°$, *so* $m\angle MQO = 45°$. *Likewise* $m\angle QMO = 45°$, *making MQO a 45°-45°-90° triangle. Therefore,* $MQ^2 = 2MO^2 \Rightarrow MQ^2 - MO^2 = 2MO^2 - MO^2 = MO^2$. *When x = 6,* $MQ^2 - MO^2 = 2 \bullet 6^2 + 6^2\sqrt{2} = \boxed{72 + 36\sqrt{2}}$.

5. *If an integer n can be expressed as the product of 4 consecutive integers, then there exists an integer x where n = x(x + 1)(x + 2)(x + 3) =* $(x^2 + 3x)(x^2 + 3x + 2) = (x^2 + 3x + 1 - 1)(x^2 + 3x + 1 + 1) = (x^2 + 3x + 1)^2 - 1$. *We need to find the smallest* $x^2 + 3x + 1$ *larger than 2017. The smallest value for x occurs when x = 44. Therefore,* $x^2 + 3x + 1 = a = \boxed{2069}$.

6. *The x-values for which* $\sin x = \frac{1}{2}$ *are* $\frac{\pi}{6}$ *and* $\frac{5\pi}{6}$ *on the interval [0,2π), which gives the sum π. The x-values for which* $\sin 2x = \frac{1}{2}$ *are* $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, *and* $\frac{17\pi}{12}$ *on the interval [0,2π), which gives a sum of 3π. The sum of solutions when n = 3, 4, 5, ... are 5π, 7π, and 9π, ... Extrapolating this pattern, we are looking for the nth odd number that is 139. Odd numbers are of form 2n – 1, and solving 2n – 1 = 139 gives n =* $\boxed{70}$.