Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have an extended 20 minutes for this event.

__________ 1. For a, an integer between –10 and 10 inclusive, determine the number of values of a such that $ax + y = 3$ and $y = \frac{x}{a} + 4$ are perpendicular.

__________ 2. The Young Triplets have an annoying habit. Whenever a question is asked of the three of them, two tell the truth and the third lies. When I asked them which of them was born first, they replied as follows:

Al: Bob was born first.
Bob: I am not the oldest.
Carl: Al is the oldest.
Which of the Young Triplets was born first?

$a =$ __________

$b =$ __________ 3. Given positive integers a and b, the units digit of b is 8, but Ralph thought it was 6 and got 4740 for the product of a and b. Natalie thought the units digit of b was 3 and got 4695 for the product. Determine the integer values a and b.

__________ 4. A certain number of dollars is completely divided into equal shares among some (more than one) children according to the following rules:

The first child receives \$1 plus one-sixth of what remains after that.
The second child receives \$2 plus one-sixth of what remains after that.
The third child receives \$3 plus one-sixth of what remains after that.
The fourth child . . . .

This process continues until all the money and all the children are exhausted. Determine exactly the original amount of money divided amongst the children.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event A

SOLUTIONS

20

1. For a, an integer between –10 and 10 inclusive, determine the number of values of a such that $ax + y = 3$ and $y = \frac{x}{a} + 4$ are perpendicular.

If the lines are perpendicular then the product of their slopes is –1. Since the slopes are negative reciprocals of one another, this occurs at all non-zero integer values of a, 20 total values in all.

Al

2. The Young Triplets have an annoying habit. Whenever a question is asked of the three of them, two tell the truth and the third lies. When I asked them which of them was born first, they replied as follows:

Al: Bob was born first.
Bob: I am not the oldest.
Carl: Al is the oldest.
Which of the Young Triplets was born first?

Al and Bob contradict one another. This means either Al or Bob is the single liar and Carl is telling the truth. Therefore, Al was born first..

$a =$ **15**

$b =$ **318**

3. Given positive integers a and b, the units digit of b is 8, but Ralph thought it was 6 and got 4740 for the product of a and b. Natalie thought the units digit of b was 3 and got 4695 for the product. Determine the integer values a and b.

Let a and b be the positive integers. Since Ralph thought the units digit of b was 6, he used b –2 in his multiplication. Therefore, a(b – 2) = 4740 ⇒ ab = 4740 + 2a. Natalie thought the units digit of b was 3, so she actually used b – 5 in her multiplication. Therefore, a(b – 5) = 4695 ⇒ ab = 4695 + 5a. Equating and solving, we obtain 4695 + 5a = 4740 +2a ⇒ 3a = 45 ⇒ a = 15. Then b = 318.

$25

4. A certain number of dollars is completely divided into equal shares among some (more than one) children according to the following rules:

The first child receives $1 plus one-sixth of what remains after that.
The second child receives $2 plus one-sixth of what remains after that.
The third child receives $3 plus one-sixth of what remains after that.
The fourth child . . . .

This process continues until all the money and all the children are exhausted. Determine exactly the original amount of money divided amongst the children.

Let A be the starting amount. The first child received $1 + \frac{A-1}{6} = \frac{A+5}{6}$. *The second child received* $2 + \frac{1}{6}\left(A - \frac{A+5}{6} - 2\right) = \frac{5A+55}{36}$. *Since these two amounts are equal,* $\frac{5A+55}{36} = \frac{A+5}{6} \Rightarrow A = 25$. *Therefore, the original amount divided amongst the children is $25.*

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

______________ 1. A parallelogram has a slant height of 3 and a base of 7. If the measurements are increased by 25%, what is the perimeter of the new figure?

______________ 2. The ratio of the perimeter of circle A and the perimeter of square B is $\frac{3}{2}$.

Determine exactly the ratio of the area of circle A to the area of square B.

______________ 3. Figure 3 shows $\triangle ABC$ and $\triangle ADE$ sharing segment $\overline{CD}$.

If $AE = 4$, $CB = 5$, and $BA = 10$, determine exactly CE.

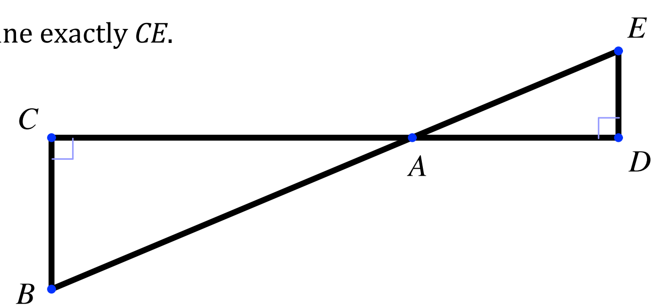

Figure 3

radius = ______________ 4. In $\triangle ABC$ *(Figure 4)*, a circle with radius r is inscribed. If $\overline{AC} \cong \overline{BC}$, $AB = 15$, $DC = 18$, and D is the midpoint of $\overline{AB}$, determine exactly the radius of the circle.

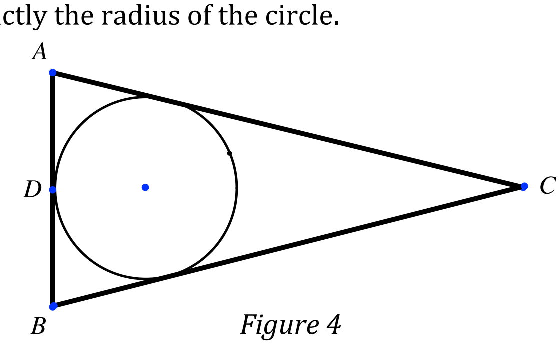

Figure 4

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event B

SOLUTIONS

25 1. A parallelogram has a slant height of 3 and a base of 7. If the measurements are increased by 25%, what is the perimeter of the new figure?

The perimeter of the original parallelogram is 20. If each measurement is increased by 25%, the perimeter of the new figure will be $20 \cdot 1.25 = 25$.

$\frac{9}{\pi}$ 2. The ratio of the perimeter of circle A and the perimeter of square B is $\frac{3}{2}$. Determine exactly the ratio of the area of circle A to the area of square B.

Let r be the length of the radius of the circle and s be the side length of the square. Since the ratio of the perimeters is $\frac{3}{2}$, $\frac{2\pi r}{4s}=\frac{3}{2} \Rightarrow \frac{r}{s}=\frac{3}{\pi}$. *The ratio of the areas of the two figures is* $\frac{\pi r^2}{s^2}=\pi\left(\frac{r}{s}\right)^2=\pi\left(\frac{3}{\pi}\right)^2=\frac{9}{\pi}$.

$\sqrt{151}$ 3. Figure 3 shows $\triangle ABC$ and $\triangle ADE$ sharing segment $\overline{CD}$. If $AE = 4$, $CB = 5$, and $BA = 10$, determine exactly CE.

Label Figure 3 as shown. By the Pythagorean Theorem, $CA=\sqrt{10^2-5^2}=\sqrt{75}=5\sqrt{3}$. *Since* $\angle CAB \cong \angle DAE$ *and* $\angle C \cong \angle D$, $\Delta CAB \sim \Delta DAE$ *by AA. By similarity,*

$$\frac{AD}{AC}=\frac{AE}{AB} \Rightarrow \frac{AD}{5\sqrt{3}}=\frac{4}{10} \Rightarrow AD=2\sqrt{3},$$ *which means* $CD = 7\sqrt{3}$.

Using Pythagorean Theorem again,

$$(2)^2+\left(7\sqrt{3}\right)^2=CE^2 \Rightarrow CE=\sqrt{4+147}=\sqrt{151}.$$

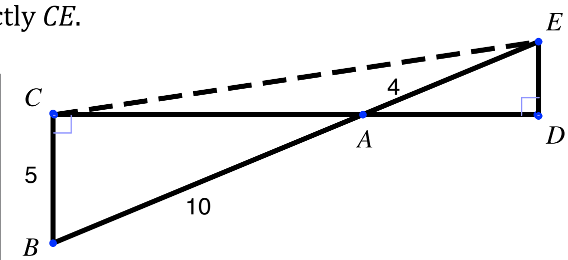

Figure 3

radius = **5** 4. In $\triangle ABC$ *(Figure 4)*, a circle with radius r is inscribed. If $\overline{AC} \cong \overline{BC}$, $AB=15$, $DC=18$, and D is the midpoint of $\overline{AB}$, determine exactly the radius of the circle.

By the Pythagorean Theorem, $AC = CA=\sqrt{7.5^2+18^2}=19.5$.

Since $\angle ACD \cong \angle ECF$, $\Delta ADC \sim \Delta FEC$ *by AA. So* $\frac{7.5}{r}=\frac{19.5}{18-r}$

$$\Rightarrow 7.5(18-r)=19.5r \Rightarrow 135-7.5r=19.5r \Rightarrow r=5.$$

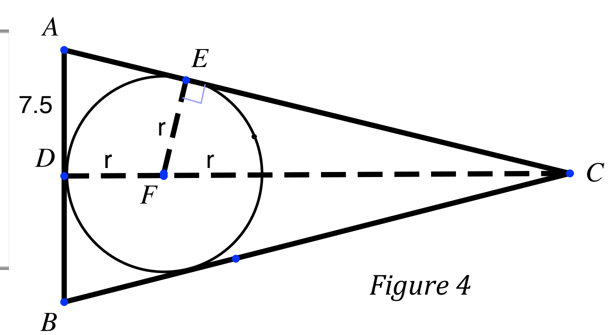

Figure 4

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

________ 1. How many ways can you arrange the letters in the word LEFT?

________ 2. Determine how many two-digit positive integers N there are such that both N and $N + 12$ have strictly increasing digits from left to right.

________ 3. Three quarters and three dimes are tossed in the air. Determine exactly the probability that the same number of quarters and dimes turn up heads.

________ 4. In Alec's class there are 155 ways to have a three-person committee consisting of either all boys or all girls. How many ways can a three-person committee be chosen that has at least one boy and at least one girl?

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event C

SOLUTIONS

24

1. How many ways can you arrange the letters in the word LEFT?

There are four letters that can come first in an arrangement, three that may come second, two that can come third, and one that can come fourth. $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ *arrangements*.

21

2. Determine how many two-digit positive integers N there are such that both N and $N + 12$ have strictly increasing digits from left to right.

The right-most digit of N must be 7 or less because adding 12 would produce a number whose right-most digit is 0 or 1 and therefore cannot be a strictly increasing number. There are ${}_7C_2 = 21$ *ways to choose two distinct positive digits from {1,2,3,4,5,6,7}. Each choice results in a valid number since we can just arrange the chosen digits in increasing order. Thus there are 21 valid two-digit positive integers.*

$\frac{5}{16}$

3. Three quarters and three dimes are tossed in the air. Determine exactly the probability that the same number of quarters and dimes turn up heads.

There are 64 ways three quarters and three dimes can be tossed. We can have 0 quarters and dimes that turn up heads or 3 quarters and 3 dimes that turn up heads. There are only two ways this can can happen. However, there are ${}_3C_1 \cdot {}_3C_1 = 9$ *ways the quarters and dimes can have one head a piece. There are* ${}_3C_2 \cdot {}_3C_2 = 9$ *ways the quarters and dimes can have two heads a piece. Therefore, there are 20 ways of obtaining the same number of heads on both sets of coins, giving us a probability of* $\frac{20}{64} = \frac{5}{16}$.

525

4. In Alec's class there are 155 ways to have a three-person committee consisting of either all boys or all girls. How many ways can a three-person committee be chosen that has at least one boy and at least one girl?

Let b be the the number of boys in the class and g be the number of girls in the class. Since there are 155 ways of picking a committee of all boys or all girls, ${}_bC_3 + {}_gC_3 = 155$. *Listing out the first several values of* ${}_nC_3$ *for n = 3, 4, 5, 6, … are 1, 4, 10, 20, 35, 56, 84, 120, 165. The only way to obtain the sum of 155 is if there are 7 boys and 10 girls or there are 10 boys and 7 girls. In either case, there are* ${}_{17}C_3 - 155 = 680 - 155 = 525$ *ways of choosing a committee with at least one boy and one girl.*

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

$a =$ ______________

1. Two of the three sides of a triangle are 17 and 19. The inequality used to describe the perimeter can be written in the form $a < p < b$. Determine exactly the maximum value of a.

____

2. The difference of two positive integers is $\frac{1}{3}$ of their sum. Determine exactly the ratio written as a fraction, of the smaller number to the larger number.

$AC =$ ______________

3. Quadrilateral $ABCD$ is inscribed in a circle with $\angle BDC = 50°$, $\angle ACB = 80°$, $AD = 5$, and $BC = 8$.Determine exactly AC.

____

4. Let a, b, and c be three distinct two-digit integers. Determine exactly the minimum value of the difference of the roots of the equation $(x + a)(x - b) + (x - c)(x + a) = 0$.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2015-16 Meet 5, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

$a = \boxed{38}$

1. Two of the three sides of a triangle are 17 and 19. The inequality used to describe the perimeter can be written in the form $a < p < b$. Determine exactly the maximum value of a.

[2015 AMC 12A, problem #2]

Let s be the third side of the triangle. Using the Triangle Inequality Theorem, 2 < s < 36, which means the perimeter inequality is 36 + 2 < p < 36 + 36 $\Rightarrow$ $38 < p < 72 \Rightarrow a = 38$.

$\boxed{\frac{1}{2}}$

2. The difference of two positive integers is $\frac{1}{3}$ of their sum. Determine exactly the ratio written as a fraction, of the smaller number to the larger number.

[2015 AMC 12A, problem #4]

Let a and b the larger and smaller numbers. Then, $a-b=\frac{1}{3}(a+b) \Rightarrow 3(a-b)=a+b \Rightarrow 2a=4b \Rightarrow \frac{b}{a}=\frac{2}{4}=\frac{1}{2}$.

$AC = \boxed{8}$

3. Quadrilateral $ABCD$ is inscribed in a circle with $\angle BDC = 50°$, $\angle ACB = 80°$, $AD = 5$, and $BC = 8$. Determine exactly AC.

[2015 AMC 12B, problem #13]

$\angle BDC$ *and* $\angle BAC$ *are both subtended by minor arc* $\overset{\frown}{BC}$, *so* $m\angle BDC = m\angle BAC = 50°$. *In* $\triangle ABC$, $m\angle ABC = 180° - (50° + 80°) = 50°$. *Therefore,* $\triangle ABC$ *is isosceles and BC = AC = 8.*

$\boxed{-196\frac{1}{2}}$ ***or*** $\boxed{-196.5}$ ***or*** $\boxed{-\frac{393}{2}}$

4. Let a, b, and c be three distinct two-digit integers. Determine exactly the minimum value of the difference of the roots of the equation $(x + a)(x - b) + (x - c)(x + a) = 0$.

[2014 AMC 12B, problem #12]

The left-hand side of the equation can be factored as $(x+a)(x-b+x-c) \Rightarrow (x+a)(2x-(b+c))$, *giving roots –a and* $\frac{b+c}{2}$. *The difference of the roots can be expressed as* $-a - \frac{b+c}{2}$ *or* $\frac{b+c}{2} + a$. *The first difference is minimized when a = 99, b = 98, and c = 97, while the second difference is minimized when a = –99, b = –98, and c = -97. Either difference has a minimum value of –196.5.*

Minnesota State High School Mathematics League

2015-16 Meet 5, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

______________ 1. At Minnepaul High, there are 3 different clubs: foosball, philately, and full contact cribbage. Each of the 101 students is in at least 2 clubs, and each club has exactly 83 members. What is the largest number of students that could be in all 3 clubs?

______________ 2. An upside down cone is 4 inches in diameter and 12 inches tall as shown in *Figure 2*. The cone is filled with liquid, but there is a hole at the apex of the cone. The water level decreases by a from the top of the cone, which results in the remaining water being half the volume. Determine exactly how much the water level decreased.

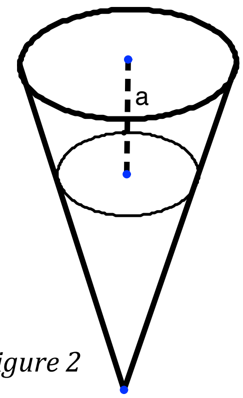

Figure 2

______________ 3. Determine exactly the number of points (a, b) in the first quadrant with integer coordinates that lie 25 units away from (8, 16).

______________ 4. How many integer solutions are there to $a + b + c + d + e = 20$ if we require $0 < a < b < c < d < e < 10$?

______________ 5. A circle has been inscribed inside of one quadrant of the unit circle and is tangent to the x-axis and y-axis. If two spheres are created, having the same radii as the two circles, determine exactly the ratio of the larger volume to the smaller volume.

______________ 6. In $\triangle ABC$, $m\angle ACB = 90°$ and $AB = 10$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. Determine exactly the perimeter of $\triangle ABC$.

Team: ______________________

Minnesota State High School Mathematics League

2015-16 Meet 5, Team Event

SOLUTIONS (page 1)

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1. At Minnepaul High, there are 3 different clubs: foosball, philately, and full contact cribbage. Each of the 101 students is in at least 2 clubs, and each club has exactly 83 members. What is the largest number of students that could be in all 3 clubs?

$12-6\sqrt[3]{4}$ *or* $12-6\bullet 2^{\frac{2}{3}}$

2. An upside down cone is 4 inches in diameter and 12 inches tall as shown in *Figure 2*. The cone is filled with liquid, but there is a hole at the apex of the cone. The water level decreases by *a* from the top of the cone, which results in the remaining water being half the volume. Determine exactly how much the water level decreased.

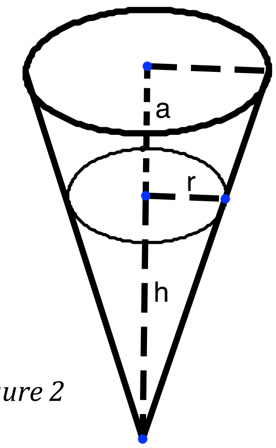

Figure 2

9

3. Determine exactly the number of points (a, b) in the first quadrant with integer coordinates that lie 25 units away from (8, 16).

6

4. How many integer solutions are there to $a + b + c + d + e = 20$ if we require $0 < a < b < c < d < e < 10$?

$7+5\sqrt{2}$

5. A circle has been inscribed inside of one quadrant of the unit circle and is tangent to the x-axis and y-axis. If two spheres are created, having the same radii as the two circles, determine exactly the ratio of the larger volume to the smaller volume.

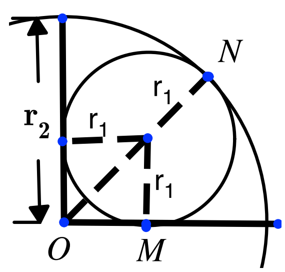

Figure 5

$10+10\sqrt{2}$

6. In $\triangle ABC$, $m\angle ACB = 90°$ and $AB = 10$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. Determine exactly the perimeter of $\triangle ABC$.

[2014 AMC 12B, problem #19]

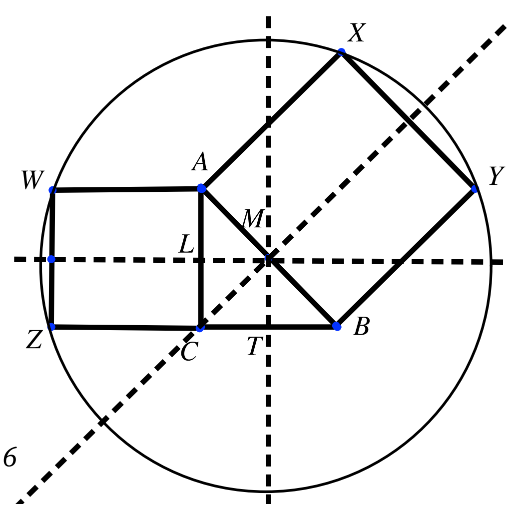

Figure 6

Minnesota State High School Mathematics League

2015-16 Meet 5, Team Event

SOLUTIONS (page 2)

1. *Let $N(F \cap P \cap C)$ be the number of students who belong to all three clubs, and let $N(F \cap P)$, $N(P \cap C)$, and $N(F \cap C)$ be the number of students who belong to pairwise intersections of those clubs. Then 101 = $3 \bullet 83 - N(F \cap P) - N(P \cap C) - N(F \cap C) + N(F \cap P \cap C) \Rightarrow N(F \cap P) + N(P \cap C) + N(F \cap C) = 148 + N(F \cap P \cap C)$. However, the left side of this equation is just $101 + 2N(F \cap P \cap C)$ since we are triple counting the students in all three clubs. Therefore,*
$101 + 2N(F \cap P \cap C) = 148 + N(F \cap P \cap C) \Rightarrow N(F \cap P \cap C) = \boxed{47.}$

2. *The original volume of the cone is $\frac{1}{3} \bullet \pi \bullet 2^2 \bullet 12 = 16\pi in^3$. The triangles formed by the cross-sections of the two cones are similar. The smaller triangle has a radius r and a height of h. Therefore, $\frac{r}{h} = \frac{2}{12} \Rightarrow h = 6r$. Since half the water has drained from the original cone, there is $8\pi in^3$ left. Thus, $\frac{1}{3} \bullet \pi \bullet r^2 \bullet 6r = 8\pi in^3 \Rightarrow r = \sqrt[3]{4}$ and*
h = $6\sqrt[3]{4}$. *The water level has decreased by* $\boxed{12 - 6\sqrt[3]{4}}$.

3. *The points all must lie on a circle centered at (8,16) with a radius of 25. A distance of 25 can be obtained by translating (8, 16) using Pythagorean Triples 7–24–25 and 15–20–25. We can translate the point right 7 and up 24, left 7 and up 24, right 24 and up 7, right 24 and down 7, right 15 and up 20, right 20 and up 15, right 20 and down 15. We may also translate (8, 16) directly up 25 or directly right 25. There are* $\boxed{9}$ *points overall.*

4. *Start by conditioning on e. If e = 9, the only solution for (a, b, c, d, e) is (1, 2, 3, 5, 9) since d can only be 5 and a, b, and c can only be 1, 2, and 3. If e = 8, we have two solutions: (1, 2, 3, 6, 8) and (1, 2, 4, 5, 8) since d can be either 5 or 6. If e = 7, there are only two solutions, namely (1, 2, 4, 6, 7) and (1, 3, 4, 5, 7). When e = 6, we get (2, 3, 4, 5, 6). Smaller values of e do not work, making* $\boxed{6}$ *solutions.*

5. *The center of the smaller circle is located along a line segment ON creating a $45° - 45° - 90°$ as shown in Figure 5. The distance from the center of the smaller circle to the origin is $r_1\sqrt{2}$, so $r_2 = r_1 + r_1\sqrt{2} = r_1(1+\sqrt{2})$. This means the ratio of the two radii is $1+\sqrt{2}$, making the ratio of two spheres created with the same larger and smaller radii*
$(1+\sqrt{2})^3 = \boxed{7+5\sqrt{2}}$.

6. *Let us find the center of this circle. The center can be found by drawing the perpendicular bisectors of segments $\overline{WZ}$ and $\overline{XY}$ as shown in Figure 6. The perpendicular bisector of $\overline{WZ}$ is also the perpendicular bisector of $\overline{AC}$. The perpendicular bisectors of the legs of a right triangle will always meet at the midpoint of the hypotenuse thus creating two smaller congruent triangles half the size of the original using AA similarity and HL congruency. This means M is center of the circle and the midpoint of $\overline{AB}$. This creates a small square LMTC, which means LM = LC = AL. Therefore, ΔALM and ΔABC are both 45°-45°-90° right triangles. Therefore, BC = AC = $\frac{10}{\sqrt{2}} = 5\sqrt{2}$ and the perimeter of ΔABC is*
$\boxed{10+10\sqrt{2}}$.