Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

x = ____________

1. Determine exactly the positive solution to $16x^2 = 100$.

$(m, n) =$ ____________

2. Given that $\dfrac{2x^3 + 14x^2 + 12x}{4x + 4} \div (mx + n) = x$, determine exactly the values of *m* and *n.*

width = ____________

3. A square patio will have a border added to only two adjacent sides, making a larger square. The border will add 6 inches to each of those two sides. Determine exactly the width in feet of the initial square patio so that the total area of the extended patio with border is $12\frac{1}{4}$ square feet.

____

4. Determine exactly the value of $\dfrac{1}{\sqrt{2}+\sqrt{1}} + \dfrac{1}{\sqrt{3}+\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{400}+\sqrt{399}}$.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

$x = \boxed{\frac{5}{2}}$ or $\boxed{2\frac{1}{2}}$ or $\boxed{2.5}$

1. Determine exactly the positive solution to $16x^2 = 100$.

$16x^2 = 100 \Rightarrow x^2 = \frac{100}{16} \Rightarrow x = \pm\frac{10}{4} = \pm\frac{5}{2}$. *The solution is positive, so* $x = \frac{5}{2}$.

$(m, n) = \boxed{\left(\frac{1}{2}, 3\right)}$

2. Given that $\frac{2x^3 + 14x^2 + 12x}{4x+4} \div (mx + n) = x$, determine exactly the values of m and n.

$\frac{2x^3 + 14x^2 + 12x}{4x+4} \div (mx+n) = x \Rightarrow \frac{2x\cancel{(x+1)}(x+6)}{4\cancel{(x+1)}} \bullet \frac{1}{mx+n} = x \Rightarrow \frac{x+6}{2(mx+n)} = 1$, *so* $(m, n) = \left(\frac{1}{2}, 3\right)$.

width = $\boxed{3\ \textit{feet}}$

3. A square patio will have a border added to only two adjacent sides, making a larger square. The border will add 6 inches to each of those two sides. Determine exactly the width in feet of the initial square patio so that the total area of the extended patio with border is $12\frac{1}{4}$ square feet.

Let x equal the side length of patio without border. The side length of the patio with the border will be $x + \frac{1}{2}$ *ft. Since the area of the patio is* $12\frac{1}{4}$ *sq. ft.*, $\left(x + \frac{1}{2}\right)^2 = 12\frac{1}{4} = \frac{49}{4} \Rightarrow x + \frac{1}{2} = \pm\frac{7}{2} \Rightarrow x = 3$ *or* $x = -4$.
Since the width is positive, x = 3 .

$\boxed{19}$

4. Determine exactly the value of $\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \ldots + \frac{1}{\sqrt{400}+\sqrt{399}}$.

We can solve this problem by rationalizing the denominator of each term. Therefore,

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \ldots + \frac{1}{\sqrt{400}+\sqrt{399}} = \left(\cancel{\sqrt{2}} - \sqrt{1}\right) + \left(\cancel{\sqrt{3}} - \cancel{\sqrt{2}}\right) + \left(\cancel{\sqrt{4}} - \cancel{\sqrt{3}}\right) \ldots + \left(\sqrt{400} - \cancel{\sqrt{399}}\right).$$

Rewriting and simplifying, we notice that all terms cancel except $\sqrt{400}$ *and* $-\sqrt{1}$. *This gives a sum of 19.*

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

Volume = __________

1. Determine exactly the volume of a sphere whose surface area is 36π.

$m\overset{\frown}{AD} =$ __________

2. In circle O, minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ each measure $73°$ and $m\angle AEB = 82°$, as shown in *Figure 2*. Determine exactly the measure of $\overset{\frown}{AD}$.

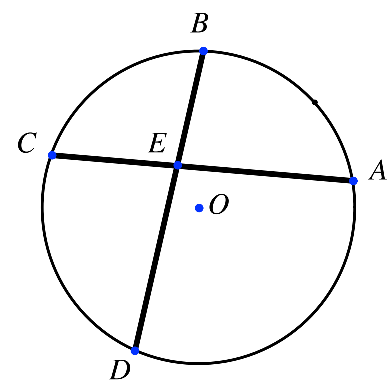

Figure 2

Area = __________

3. Determine exactly the area of the region of points that satisfy both $x^2 + y^2 \le 36$ and $y \ge 3$.

$XY =$ __________

4. Equilateral triangle ABC has sides of length 7. Circle O is drawn so that it is tangent to two sides of ABC, as shown in *Figure 4*. If $m\angle EYX = 90°$, calculate the length of chord $\overline{XY}$.

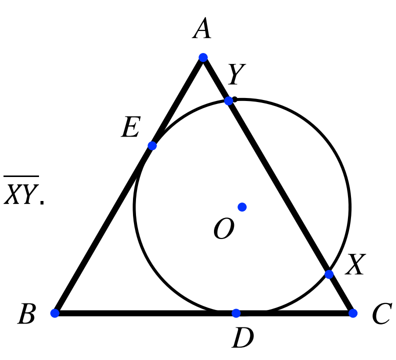

Figure 4

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event B

SOLUTIONS

$\boxed{36\pi}$

1. Determine exactly the volume of a sphere whose surface area is 36π.

> $4\pi r^2 = 36\pi \Rightarrow r = 3$. *The volume is then* $\frac{4}{3}\pi r^3 = 36\pi$.

$m\overset{\frown}{AD} = \boxed{123^\circ}$

2. In circle O, minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ each measure 73° and $m\angle AEB = 82^\circ$, as shown in *Figure 2*. Determine exactly the measure of $\overset{\frown}{AD}$.

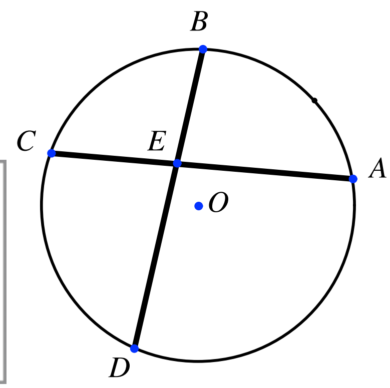

Figure 2

> *Intersecting chords* $\overline{BD}$ *and* $\overline{CA}$ *cut off arcs* $\overset{\frown}{AB}$ *and* $\overset{\frown}{CD}$, *with the relationship that* $m\angle AEB = \frac{1}{2}\left(m\overset{\frown}{AB} + m\overset{\frown}{CD}\right)$. *Therefore,* $m\overset{\frown}{AB} + m\overset{\frown}{CD} = 2\bullet 82^\circ = 164^\circ$ *and* $m\overset{\frown}{AD} = 360^\circ - 164^\circ - 73^\circ = 123^\circ$.

$\boxed{12\pi - 9\sqrt{3}}$

3. Determine exactly the area of the region of points that satisfy both $x^2 + y^2 \le 36$ and $y \ge 3$.

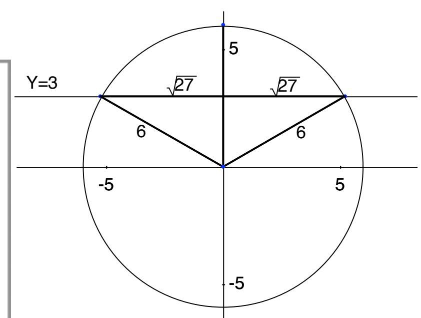

Figure 3

> *The area we seek lies inside the circle and above the line y = 3. Each small triangle has angle measures of 30°-60°-90°. Find the area of the big wedge formed by the angle range* $30^\circ \le \theta \le 150^\circ$. *This wedge has an area of* $\frac{1}{3}\bullet\pi\bullet 6^2 = 12\pi$. *The "big triangle" has a base length of* $2\sqrt{27} = 6\sqrt{3}$, *a height of 3, and an area of* $\frac{1}{2}\bullet 3\bullet 6\sqrt{3} = 9\sqrt{3}$. *The area of the region is* $12\pi - 9\sqrt{3}$.

$XY = \boxed{\frac{21}{5}}$ ***or*** $\boxed{4\frac{1}{5}}$ ***or*** $\boxed{4.2}$

4. Equilateral triangle ABC has sides of length 7. Circle O is drawn so that it is tangent to two sides of ABC, as shown in *Figure 4*. If $m\angle EYX = 90^\circ$, calculate the length of chord $\overline{XY}$.

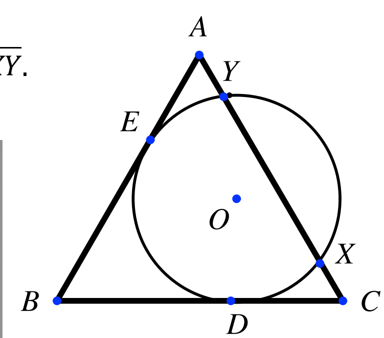

Figure 4

> *Let BD = x. Since tangents to a circle from the same point are congruent, BE = BD. Therefore, triangle BDE is equilateral. This means* $m\angle YED = 90^\circ$. *EDXY is a rectangle, so XY = ED = x. CX = AY =* $\frac{7-x}{2}$. *Using the Power of a Point with respect to C and solving ,* $(7-x)^2 = \left(\frac{7-x}{2}\right)\left(\frac{7+x}{2}\right) \Rightarrow 7 - x = \frac{7+x}{4} \Rightarrow x = \frac{21}{5}$.

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

_______________ 1. What is the nth term of the geometric sequence whose first three terms are $\frac{3}{2}$, 3, and 6 ?

$b =$ _______________ 2. Determine exactly all values of b such that $\sum_{x=1}^{5}\left(x+b^2\right)=60$.

$f\left(n\right)=$ _______________ 3. For a design, Margo creates n concentric circles out of yarn. The first and smallest circle has a radius of 2 cm. Each circle after that has a radius that is 1 cm greater than the previous circle. For gluing purposes she needs .5 cm of extra yarn for each circle. If $f\left(n\right)$ represents the total amount of yarn needed for n circles, express $f\left(n\right)$ in terms of $f\left(n-1\right)$ and n.

$a_1 =$ _______________ 4. Given a geometric sequence with terms $a_{x-1}=32,\ \ldots\ a_{x+2}=108,\ \ldots\ a_{2x}=243$, where x is a positive integer, determine exactly the value of a_1 .

Name: _________________________ Team: _________________________

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$g_n = \frac{3}{2} \bullet 2^{n-1}$ or $g_n = 3 \bullet 2^{n-2}$

1. What is the nth term of the geometric sequence whose first three terms are $\frac{3}{2}$, 3, and 6 ?

The explicit formula for a geometric sequence is $g_n = g_1 r^{n-1}$. *Here, the common ratio is two; therefore,* $g_n = \frac{3}{2} \bullet 2^{n-1}$.

$b = 3$ or -3

2. Determine exactly all values of b such that $\sum_{x=1}^{5}\left(x + b^2\right) = 60$.

Rewrite the given equation using the properties of partial sums to get
$\sum_{x=1}^{5}\left(x + b^2\right) = 60 \Rightarrow \sum_{x=1}^{5} x + \sum_{x=1}^{5} b^2 = 60 \Rightarrow 15 + 5b^2 = 60 \Rightarrow b = \pm 3.$

$f(n) = f(n-1) + (2n+2)\pi + \frac{1}{2}.$

3. For a design, Margo creates n concentric circles out of yarn. The first and smallest circle has a radius of 2 cm. Each circle after that has a radius that is 1 cm greater than the previous circle. For gluing purposes she needs .5 cm of extra yarn for each circle. If $f(n)$ represents the total amount of yarn needed for n circles, express $f(n)$ in terms of $f(n-1)$ and n.

The amount of yarn needed for the first circle, f(1), is $4\pi + \frac{1}{2}$. *The total amount of yarn needed for two concentric circles, f(2), is f(1) +* $6\pi + \frac{1}{2}$. *Notice, the nth concentric circle adds* $(2n+2)\pi + \frac{1}{2}$ *cm to the total amount of yarn. Therefore, f(n) = f(n−1) +* $(2n+2)\pi + \frac{1}{2}$.

$a_1 = \frac{128}{9}$

4. Given a geometric sequence with terms $a_{x-1} = 32, \ldots\ a_{x+2} = 108, \ldots\ a_{2x} = 243$, where x is a positive integer, determine exactly the value of a_1.

Using the explicit formula for a geometric sequence, we obtain $32 = a_1 \cdot r^{x-2}$, $108 = a_1 \cdot r^{x+1}$, $243 = a_1 \cdot r^{2x-1}$. *Dividing second equation by the first equation gives* $108 = 32 \cdot r^3 \Rightarrow r^3 = \frac{27}{8} \Rightarrow r = \frac{3}{2}$. *Notice,*
$108 r^{x-2} = a_1 \cdot r^{2x-1} = 243 \Rightarrow 108\left(\frac{3}{2}\right)^{x-2} = 243 \Rightarrow \left(\frac{3}{2}\right)^{x-2} = \frac{9}{4} = \left(\frac{3}{2}\right)^2 \Rightarrow x - 2 = 2 \Rightarrow x = 4.$ *Substituting r and x values back into our first equation gives us* $32 = a_1 \cdot \left(\frac{3}{2}\right)^{4-2} \Rightarrow 32 = a_1 \cdot \frac{9}{4} \Rightarrow a_1 = \frac{128}{9}.$

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Determine exactly the focal point of $\frac{x^2}{16} - \frac{y^2}{16} = 1$ that lies on the positive x-axis.

________________ 2. Write in vertex form the equation of a parabola with a vertex of (4,2) and a focus of (4,0).

________________ 3. There are 30 apple trees in an orchard and each produces an average of 400 apples per year. For each additional tree added to the orchard, the number of apples produced by each tree drops by 10 since there is more competition for resources. Calculate the maximum number of apples that the orchard can produce if more trees are planted.

r = ________________ 4. The center of circle *O* lies on the y-axis. If circle *O* is tangent to the x-axis and to the curve $x^2 - y^2 = 4$, determine exactly the radius of circle *O*.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2015-16 Meet 4, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\left(4\sqrt{2},0\right)}$

1. Determine exactly the focal point of $\frac{x^2}{16}-\frac{y^2}{16}=1$ that lies on the positive x-axis.

The center of the hyperbola is (0,0) with its x-intercepts a distance of 4 from the center. The foci are c units from the center. Therefore, $a^2+b^2=c^2 \Rightarrow 4^2+4^2=c^2 \Rightarrow c=\pm\sqrt{32}=\pm4\sqrt{2}$. *The focal point on the positive x-axis is* $\left(4\sqrt{2},0\right)$.

$\boxed{y=\frac{-1}{8}\left(x-4\right)^2+2}$

2. Write in vertex form the equation of a parabola with a vertex of (4,2) and a focus of (4,0).

A parabola with the vertical axis of symmetry can be written in the form $\left(x-h\right)^2=4p\left(y-k\right)$, *where (h,k) is the vertex and p is the directed distance from the directrix. Substituting and rearranging we obtain* $\left(x-4\right)^2=4\bullet-2\left(y-2\right) \Rightarrow y=\frac{-1}{8}\left(x-4\right)^2+2$.

$\boxed{12{,}250}$

3. There are 30 apple trees in an orchard and each produces an average of 400 apples per year. For each additional tree added to the orchard, the number of apples produced by each tree drops by 10 since there is more competition for resources. Calculate the maximum number of apples that the orchard can produce if more trees are planted.

Let x = the number of added trees. Then the total number of apples produced is $T\left(x\right)=\left(30+x\right)\left(400-10x\right)$. *The maximum number of apples will occur at the midpoint of the x-intercepts of the graph of T(x), namely (5,0) since the intercepts are (–30,0) and (40,0). The maximum number of apples produced is 35 · 350 = 12,250 apples.*

$r=\boxed{2\sqrt{2}}$ ***or*** $\boxed{\sqrt{8}}$

4. The center of circle *O* lies on the y-axis. If circle *O* is tangent to the x-axis and to the curve $x^2-y^2=4$, determine exactly the radius of circle *O*.

Let r = the radius of circle O. If the circle's center lies on the y-axis and the circle is tangent to the x-axis, then the equation of the circle is: $x^2+\left(y-r\right)^2=r^2$. *Subtracting* $x^2-y^2=4$ *from* $x^2+\left(y-r\right)^2=r^2$ *yields* $2y^2-2ry=-4 \Rightarrow y^2-ry+2=0$. *Since these curves are tangent, the discriminant is equal to 0. Therefore,* $r^2-8=0 \Rightarrow r=\pm\sqrt{8}=\pm2\sqrt{2}$. *r is a radius, so r =* $2\sqrt{2}$.

Minnesota State High School Mathematics League

2015-16 Meet 4, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

______________ 1. Given a set of functions where $f_1(x)=7-x$, $f_2(x)=f_1(x)+1$, and for $n>2$, $f_n(x)=3f_{n-2}(x)-f_{n-1}(x)$. For a particular x = a, $f_3(a)=f_2(a)+1$. Determine the least value of n such that $f_n(a)<0$.

______________ 2. Determine exactly the largest possible surface area of a cone with base radius of 15 that will fit inside a sphere of radius 17.

______________ 3. The graphs of $y=\frac{x^2}{4p}$ and $\frac{x^2}{16}+\frac{y^2}{27}=1$ intersect at points A and B. Let O be the origin. If $\triangle AOB$ is equilateral, determine exactly the positive value for p.

$x=$ ______________ 4. Given an arithmetic sequence with terms $a_x=12,\ \ldots\ a_{2x+1}=32,\ \ldots\ a_{4x-1}=52,$ where x is a positive integer. Determine exactly the value of x.

$|b|=$ ______________ 5. Let $f(x)=\frac{1}{x-1}$ and the notation $f^n(x)$ denote f applied repeatedly. For example, $f^3(x)=f(f(f(x)))$. The function $f^8(x)$ will be a rational expression of the form $\frac{ax+b}{cx+d}$. Determine exactly the absolute value of b.

$(x,y)=$ ______________ 6. Determine exactly the focal point of $xy=1$ that lies in the first quadrant.

Team: ______________________

Minnesota State High School Mathematics League

2015-16 Meet 4, Team Event

SOLUTIONS (page 1)

11

1. Given a set of functions where $f_1(x)=7-x$, $f_2(x)=f_1(x)+1$, and for $n>2$, $f_n(x)=3f_{n-2}(x)-f_{n-1}(x)$. For a particular x = a, $f_3(a)=f_2(a)+1$. Determine the least value of n such that $f_n(a)<0$.

$225\pi+75\pi\sqrt{34}$

2. Determine exactly the largest possible surface area of a cone with base radius of 15 that will fit inside a sphere of radius 17.

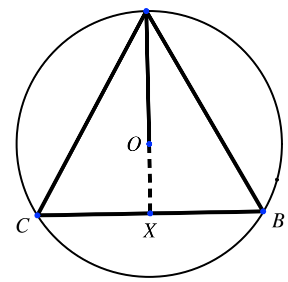

Figure 2

$\frac{\sqrt{3}}{5}$

3. The graphs of $y=\frac{x^2}{4p}$ and $\frac{x^2}{16}+\frac{y^2}{27}=1$ intersect at points A and B. Let O be the origin. If $\triangle AOB$ is equilateral, determine exactly the positive value for p.

3

4. Given an arithmetic sequence with terms $a_x=12,\ \ldots\ a_{2x+1}=32,\ \ldots\ a_{4x-1}=52,$ where x is a positive integer. Determine exactly the value of x.

$|b|=$ **21**

5. Let $f(x)=\frac{1}{x-1}$ and the notation $f^n(x)$ denote *f* applied repeatedly. For example, $f^3(x)=f(f(f(x)))$. The function $f^8(x)$ will be a rational expression of the form $\frac{ax+b}{cx+d}$. Determine exactly the <u>absolute</u> <u>value</u> of *b*.

$(x,y)=$ **$(\sqrt{2},\sqrt{2})$**

6. Determine exactly the focal point of *xy* = 1 that lies in the first quadrant.

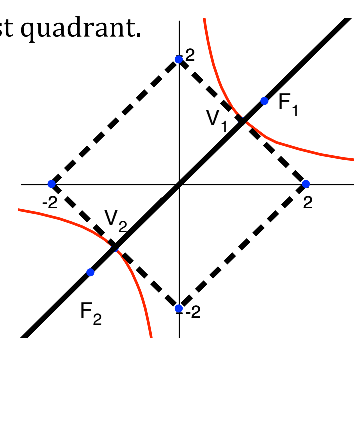

Figure 6

Minnesota State High School Mathematics League

2015-16 Meet 4, Team Event

SOLUTIONS (page 2)

1. $f_2(x)=(7-x)+1=8-x$. *Also,* $f_3(x)=(8-x)+1=9-x$. *But,* $f_3(x)=3f_1(x)-f_2(x)=3(7-x)-(8-x)=13-2x$. *Setting 9 – x = 13 – 2x gives x = 4. Thus, we obtain the following sequence:* $f_1(4)=3, f_2(4)=4, f_3(4)=5,$ $f_4(4)=3\cdot4-5=7, f_5(4)=3\cdot5-7=8, f_6(4)=3\cdot7-8=13, f_7(4)=3\cdot8-13=11, f_8(4)=3\cdot13-11=28,$ $f_9(4)=3\cdot11-28=5, f_{10}(4)=3\cdot28-5=79,$ *and* $f_{11}(4)=3\cdot5-79=-64$. *Therefore, n =* $\boxed{11}$.

2. *Let h equal the height of the cone. We can draw a vertical cross-section of the cone that cuts through the center of the sphere and the cone, as shown in Figure 2.* $\overline{BC}$ *is the diameter of the base of the cone, so BC = 30. Since O is the center of the sphere, OA = OB = OC = 17. Using the Pythagorean Theorem on triangle OBX, we get* $OX=\sqrt{17^2-15^2}=8$. *Therefore, the height of the cone is 17 + 8 = 25, so the lateral area of the cone is* $\pi rl=\pi r\cdot AB=\pi\cdot15\sqrt{25^2+15^2}=75\pi\sqrt{34}$. *The lateral area plus the area of the base gives a surface area of* $\boxed{225\pi+75\pi\sqrt{34}.}$.

3. *Let B be the point of intersection in the first quadrant. Since B is a point on an upside down equilateral triangle, we can parameterize B as* $(t,t\sqrt{3})$. *Since B is on the ellipse,* $\frac{t^2}{16}+\frac{3t^2}{27}=1\Rightarrow t^2=\frac{144}{25}\Rightarrow t=\frac{12}{5}$. *B is also on the curve* $y=\frac{x^2}{4p}$. *Substituting B into* $y=\frac{x^2}{4p}$ *yields* $t\sqrt{3}=\frac{t^2}{4p}\Rightarrow t=4p\sqrt{3}\Rightarrow\frac{12}{5}=4p\sqrt{3}$. *Then, p =* $\boxed{\frac{\sqrt{3}}{5}}$.

4. *Using the explicit formula for an arithmetic sequence, we obtain* $12=a_1+d(x-1), 32=a_1+d(2x), 52=a_1+d(4x-2)$. *Solving for* a_1 *in the second equation and substituting into the first and third equations respectively gives* $(32-2dx)+d(x-1)=12\Rightarrow 20=d(x+1)$ *and* $(32-2dx)+d(4x-2)=52\Rightarrow d(2x-2)=20$. *Notice,* both equation equal to 20, so $d(x+1)=d(2x-2)\Rightarrow x+1=2x-2\Rightarrow x=\boxed{3}$.

5. *Start completing one iteration at a time. Since* $f(x)=\frac{1}{x-1}$, $f^2(x)=\frac{1}{\frac{1}{x-1}-1}=\frac{x-1}{2-x}$. *Continuing this process we get* $f^3(x)=\frac{-x+2}{2x-3}, f^4(x)=\frac{2x-3}{-3x+5}, f^5(x)=\frac{-3x+5}{5x-8}, f^6(x)=\frac{5x-8}{-8x+13}$. *Notice the absolute value of the constant terms in the numerator form the Fibonacci sequence, so the absolute value of the b term in the numerator for* $f^8(x)$ *is* $\boxed{21}$.

6. *The curve xy = 1 is a hyperbola rotated* 45°. *The distance between vertices (1,1) and (–1,–1) is* $2\sqrt{2}$. *The other side of the rectangle is formed by tangents to the hyperbola that also have a length* $2\sqrt{2}$, *as shown in Figure 6. If the hyperbola were of the form* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *both a and b would be* $\sqrt{2}$. *Thus* $c^2=a^2+b^2\Rightarrow c^2=4\Rightarrow c=2$. *If that segment length were rotated* 45° *counterclockwise, the focal point would occur at* $\boxed{(\sqrt{2},\sqrt{2})}$.