

Minnesota State High School Mathematics League

2015-16 Meet 3, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

_____ 1. Determine exactly the area of the region in the third quadrant for which $x + 3y + 15 > 0$.

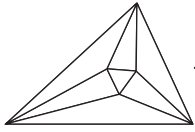
_____ 2. Last summer Alec earned \$13 for each lawn he mowed. This summer he got a raise and earned \$15 per lawn. Alec also mowed exactly twice as many lawns this summer. If he earned exactly \$187 more this summer than last, how many lawns did he mow this summer?

_____ 3. Determine exactly all values of the real number c for which the determinant of
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
 is positive.

_____ 4. In the Nelson family, there are four children. This year, they are all teenagers, and the sum of their ages is 59. In 3 years, the sum of the ages of the teenage children only will be 51. Two years after that, that sum will be only 37. What was the “teenage sum” two years ago?

Name: _____

Team: _____



Minnesota State High School Mathematics League

2015-16 Meet 3, Individual Event A

SOLUTIONS

$$\frac{75}{2} \text{ or } 37\frac{1}{2}$$

$$\text{or } 37.5$$

1. Determine exactly the area of the region in the third quadrant for which $x + 3y + 15 > 0$.

The shaded region shown in Figure 1 is a triangle whose area is $\frac{1}{2} \cdot 15 \cdot 5 = \frac{75}{2}$.

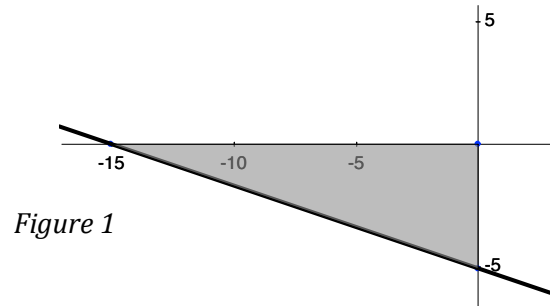


Figure 1

22

2. Last summer Alec earned \$13 for each lawn he mowed. This summer he got a raise and earned \$15 per lawn. Alec also mowed exactly twice as many lawns this summer. If he earned exactly \$187 more this summer than last, how many lawns did he mow this summer?

Let $L = \#$ of lawns Alec mowed last summer. The $\#$ of lawns Alec mowed this summer is then $2L$. Since he earned \$187 more this summer than last, $15(2L) - 13L = 187 \Rightarrow 17L = 187 \Rightarrow L = 11$. Therefore, Alec mowed 22 lawns this summer.

$$c < 5$$

3. Determine exactly all values of the real number c for which the determinant of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ is positive.}$$

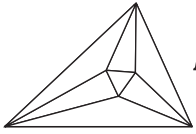
The determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & c & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is positive when

$$1(9c - 48) - 2(4 \cdot 9 - 7 \cdot 6) + 3(32 - 7c) > 0 \Rightarrow 60 - 12c > 0 \Rightarrow c < 5.$$

28

4. In the Nelson family, there are four children. This year, they are all teenagers, and the sum of their ages is 59. In 3 years, the sum of the ages of the teenage children only will be 51. Two years after that, that sum will be only 37. What was the "teenage sum" two years ago?

Let the current ages of the children be $a \leq b \leq c \leq d$. Then $a + b + c + d = 59$, and $(a + 3) + (b + 3) + (c + 3) = 51 \Rightarrow a + b + c = 42 \Rightarrow d = 17$, since the oldest child will not be a teenager in 3 years. In 5 years, $(a + 5) + (b + 5) = 37 \Rightarrow c = 15$. This implies that $a + b = 27$, which means the only way a and b can be ages of teenagers is if $a = 13$ and $b = 14$. Therefore, the sum of the teenaged children 2 years ago would be $13 + 15 = 28$.



Minnesota State High School Mathematics League

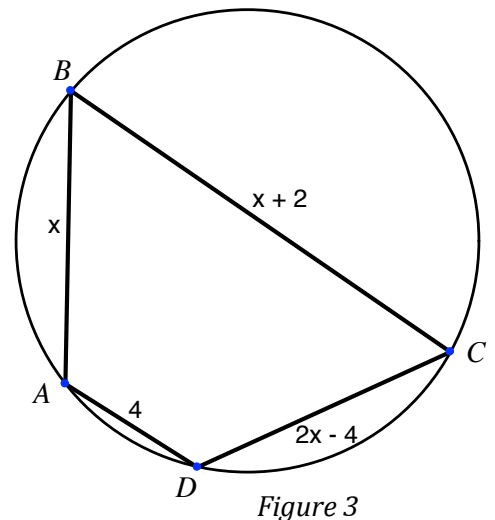
2015-16 Meet 3, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

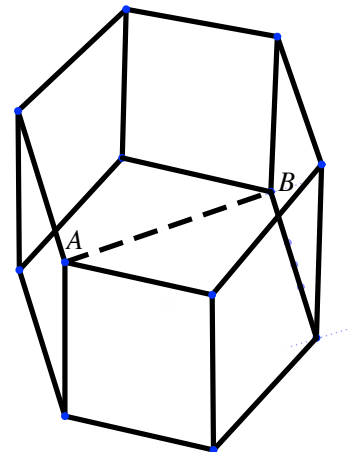
_____ 1. If the surface area and volume of a cube are equal, determine exactly the length of each of the cube's edges.

_____ 2. The volume of a cone is twice the volume of a cylinder. If the bases of the cone and cylinder are the same and the height of the cone is 6 inches, determine exactly the height of the cylinder.

_____ 3. Cyclic quadrilateral $ABCD$ is shown in *Figure 3*. If $AC = 6.25$ and $BD = 6.4$, determine exactly the perimeter of quadrilateral $ABCD$.

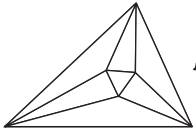


$AB =$ _____ 4. Shown in *Figure 4* is a right hexagonal prism with a surface area of $108\sqrt{3} + 288$. If bases are regular hexagons, each with area $54\sqrt{3}$, determine exactly the length of diagonal \overline{AB} .



Name: _____

Team: _____



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2015-16 Meet 3, Individual Event B

SOLUTIONS

- 6 1. If the surface area and volume of a cube are equal, determine exactly the length of each of the cube's edges.

Let each edge of the cube have length x . Then, $6x^2 = x^3 \Rightarrow x = 6$.

- 1 2. The volume of a cone is twice the volume of a cylinder. If the bases of the cone and cylinder are the same and the height of the cone is 6 inches, determine exactly the height of the cylinder.

Suppose the base of the cone and the cylinder has an area of B . The volume of the cone is $\frac{1}{3} \cdot B \cdot 6 = 2B$. The volume of the cone is twice the volume of the cylinder, so the volume of the cone is $2Bh$. Equating these volumes, we get $2B = 2Bh \Rightarrow h = 1$.

- 18 3. Cyclic quadrilateral $ABCD$ is shown in Figure 3. If $AC = 6.25$ and $BD = 6.4$, determine exactly the perimeter of quadrilateral $ABCD$.

See Figure 3. Using Ptolemy's Theorem,
 $x(2x - 4) + 4(x + 2) = 6.25 \cdot 6.4 \Rightarrow 2x^2 + 8 = 40 \Rightarrow 2x^2 - 32 = 0$. Solving this quadratic we obtain $x = -4$ (impossible) or $x = 4$. This gives us side lengths of 4, 6, 4, and 4. Therefore, the perimeter is 18.

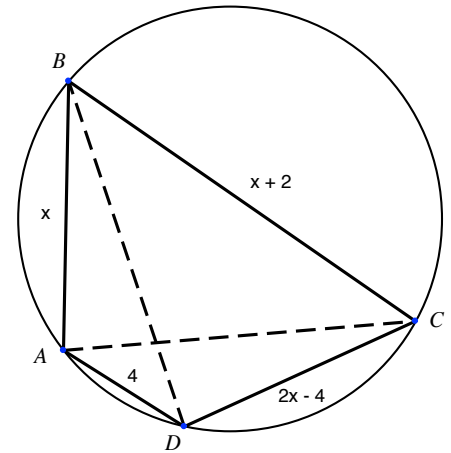


Figure 3

- $AB = 4\sqrt{13}$ 4. Shown in Figure 4 is a right hexagonal prism with a surface area of $108\sqrt{3} + 288$. If bases are regular hexagons, each with area $54\sqrt{3}$, determine exactly the length of diagonal \overline{AB} .

The base of the figure is a hexagon, which can be broken up into 6 equilateral triangles, as shown in Figure 4. Since the area of the base is $54\sqrt{3}$, each triangle has an area of $9\sqrt{3}$ and a side length of 6. This means $CB = 12$. The lateral area of the figure is 288, so each rectangular face has an area of 48 and a height of 8. Using the Pythagorean Theorem, $AB = \sqrt{8^2 + 12^2} = \sqrt{64 + 144} = \sqrt{208} = 4\sqrt{13}$.

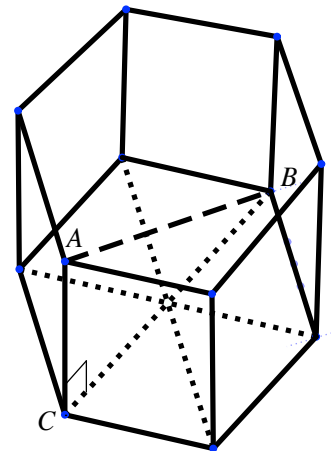
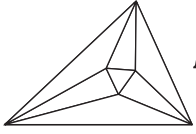


Figure 4



Minnesota State High School Mathematics League

2015-16 Meet 3, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

- _____ 1. Determine exactly the smallest positive angle measure, in radians, made with the positive x-axis by a segment connecting the point $\sqrt{1+i\sqrt{3}}$ with the origin.

- CE = _____ 2. In triangle ABC, $AB = 4$, $BC = 5$, $AC = 6$, \overline{BDE} bisects $\angle ABC$, and $\overline{CE} \perp \overline{BE}$. Determine exactly the length of \overline{CE} .

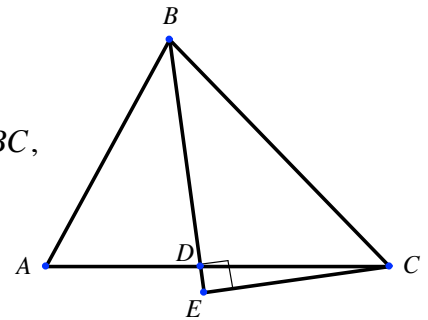


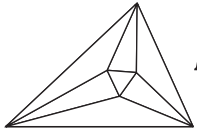
Figure 2

- _____ 3. Determine exactly the area bounded by the graph of $y = \text{Arcsin}(\sin x)$ and the x-axis, where $0 \leq x \leq 2016\pi$.

- A = _____ 4. Determine all values of A, if $\cos A - \cos B = -\sin 80^\circ$ and $A + B = 60^\circ$, where $0^\circ \leq A \leq 180^\circ$.

Name: _____

Team: _____



Minnesota State High School Mathematics League

2015-16 Meet 3, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$$\boxed{\frac{\pi}{6}}$$

1. Determine exactly the smallest positive angle measure, in radians, made with the positive x-axis by a segment connecting the point $\sqrt{1+i\sqrt{3}}$ with the origin.

By DeMoivre's Theorem, $\sqrt{1+i\sqrt{3}} = \left(2 \cdot \cos \frac{\pi}{3} + 2i \cdot \sin \frac{\pi}{3}\right)^{\frac{1}{2}} = \sqrt{2} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$. The smallest positive angle is $\frac{\pi}{6}$.

$$CE = \boxed{\frac{5\sqrt{7}}{4}}$$

2. In triangle ABC, AB = 4, BC = 5, AC = 6, \overline{BDE} bisects $\angle ABC$, and $\overline{CE} \perp \overline{BE}$. Determine exactly the length of \overline{CE} .

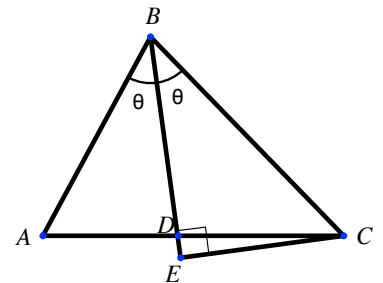


Figure 2

Let $m\angle ABC = 2\theta$. By the Law of Cosines, $6^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{8}$.

$\cos 2\theta = 2 \cos^2 \theta - 1 = \frac{1}{8} \Rightarrow \cos \theta = \frac{3}{4}$. Applying the Pythagorean Identity we obtain

$\sin \theta = \frac{\sqrt{7}}{4}$. Then $\frac{CE}{5} = \frac{\sqrt{7}}{4} \Rightarrow CE = \frac{5\sqrt{7}}{4}$.

$$\boxed{504\pi^2}$$

3. Determine exactly the area bounded by the graph of $y = \text{Arcsin}(\sin x)$ and the x-axis, where $0 \leq x \leq 2016\pi$.

The graph of $y = \text{Arcsin}(\sin x)$ is shown in Figure 3. The area of each triangular piece is equal to $\frac{1}{2} \cdot \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$ and there are 2016 triangles so the area is $\frac{2016\pi^2}{4} = 504\pi^2$.

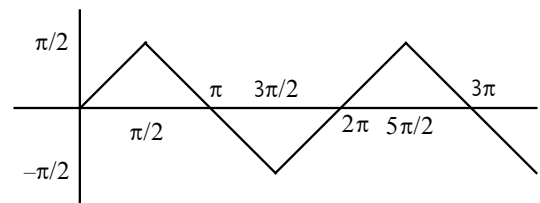


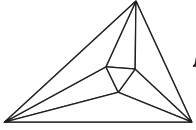
Figure 3

$$A = \boxed{110^\circ}$$

4. Determine all values of A, if $\cos A - \cos B = -\sin 80^\circ$ and $A + B = 60^\circ$, where $0^\circ \leq A \leq 180^\circ$.

or $\boxed{130^\circ}$

Substitute $60^\circ - A$ for B to obtain $\cos A - \cos(60^\circ - A) = \cos A - (\cos 60^\circ \cos A + \sin 60^\circ \sin A) = \frac{1}{2} \cos A - \frac{\sqrt{3}}{2} \sin A$. Therefore, $\frac{1}{2} \cos A - \frac{\sqrt{3}}{2} \sin A = -\sin 80^\circ \Rightarrow \frac{\sqrt{3}}{2} \sin A - \frac{1}{2} \cos A = \sin 80^\circ$. Notice the left side of this equation is simply $\sin(A - 30^\circ)$. So $\sin(A - 30^\circ) = \sin 80^\circ$. Thus, $A - 30^\circ = 80^\circ$ or $A - 30^\circ = 100^\circ$, since $\sin 80^\circ = \sin 100^\circ$. This means $A = 110^\circ$ or $A = 130^\circ$.



Minnesota State High School Mathematics League

2015-16 Meet 3, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

_____ 1. Determine exactly the value of $\left(\frac{1}{2}\right)^{-4} + \left(\frac{-1}{2}\right)^{-5}$.

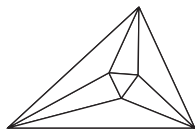
 $x =$ _____ 2. Determine exactly all values x for which $\log_2 x + \log_2(x-7) = 3$.

_____ 3. Find the largest integer b for which $\sqrt[b]{64^{321}}$ is rational.

_____ 4. Determine exactly the x value where the graphs of $y = \log_3(x)$ and $y = \log_{\frac{1}{9}}(x) + 12$ intersect.

Name: _____

Team: _____



Minnesota State High School Mathematics League

2015-16 Meet 3, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

-16

1. Determine exactly the value of $\left(\frac{1}{2}\right)^{-4} + \left(\frac{-1}{2}\right)^{-5}$.

$$\left(\frac{1}{2}\right)^{-4} + \left(\frac{-1}{2}\right)^{-5} = 2^4 + (-2)^5 = 16 - 32 = -16.$$

x = 8

2. Determine exactly all values x for which $\log_2 x + \log_2(x-7) = 3$.

Graders: deduct 1 point for any additional solution (up to 2 points).

Using the product property of logarithms, we can write our given equation as $\log_2 x(x-7) = 3$. Converting $\log_2 x(x-7) = 3$ into exponential form gives us $x(x-7) = 8 \Rightarrow x^2 - 7x = 8 \Rightarrow x^2 - 7x - 8 = 0 \Rightarrow (x-8)(x+1) = 0 \Rightarrow x = 8$ or $x = -1$. But be Careful, $x = -1$ is not possible and is therefore extraneous. Thus, $x = 8$ is the only solution.

1926

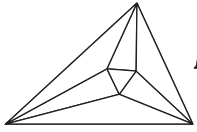
3. Find the largest integer b for which $\sqrt[b]{64^{321}}$ is rational.

First, write $\sqrt[b]{64^{321}}$ in exponential form to obtain $64^{\frac{321}{b}}$. The temptation is to say the largest integer for which $\sqrt[b]{64^{321}}$ is rational is 321, but that is incorrect. Notice, $64^{\frac{321}{b}} = (2^6)^{\frac{321}{b}} = 2^{\frac{1926}{b}}$. Therefore, the largest integer b that makes the expression rational is 1926.

6561

4. Determine exactly the x value where the graphs of $y = \log_3(x)$ and $y = \log_{\frac{1}{9}}(x) + 12$ intersect.

These graphs intersect when $\log_{\frac{1}{9}}(x) + 12 = \log_3(x) \Rightarrow \log_3(x) - \log_{\frac{1}{9}}(x) - 12 = 0$. Using the change of base, $\log_3(x) - \log_{\frac{1}{9}}(x) - 12 = 0 \Rightarrow \log_3(x) - \frac{\log_3(x)}{\log_3\left(\frac{1}{9}\right)} - 12 = 0$. Let $w = \log_3(x)$. Then we have $w + \frac{1}{2}w - 12 = 0 \Rightarrow \frac{3}{2}w = 12 \Rightarrow w = 8$, so $w = \log_3(x) = 8$. Therefore, $x = 3^8 = 6561$.



Minnesota State High School Mathematics League

2015-16 Meet 3, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

tan α =

1. Determine exactly the largest value for tan α, given the system has no solutions (x, y, z).

$$\begin{cases} 1 = \frac{x}{2} + \frac{y}{2} + z \cdot \sin \alpha \\ 1 = \frac{y}{3} + \frac{z}{3} + x \cdot \cos \alpha \\ 1 = z + x + y \end{cases}$$

Area =

2. A regular octagon with an apothem length OM of 4.83 units is inscribed inside a circle as shown in *Figure 2*. Compute the area of quadrilateral $ABCF$.

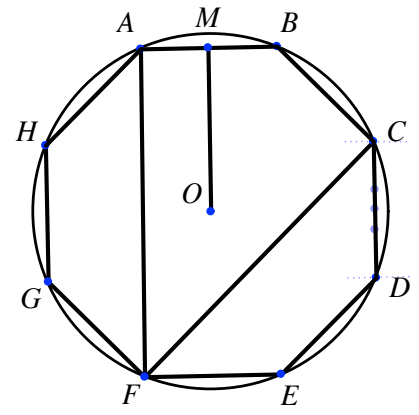


Figure 2

3. Determine exactly the value of $\frac{2 \tan \frac{\pi}{24}}{1 + \tan^2 \frac{\pi}{24}}$.

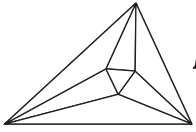
4. How many integers are in the domain of $y = \log_x \log_x \log_x (2016 - x)$?

Area =

5. Determine exactly the area of the region defined by $|x + 1| < 2$ and $|y| < |x|$.

6. If $\sin 2x = 2(\sin x + \cos x)$, determine exactly all values of $(\sin x)(\cos x)$.

Team: _____



Minnesota State High School Mathematics League

2015-16 Meet 3, Team Event

SOLUTIONS (page 1)

$$\tan \alpha = \boxed{2\sqrt{2}}$$

1. Determine exactly the largest value for $\tan \alpha$, given the system
has no solutions (x, y, z) .

$$\begin{cases} 1 = \frac{x}{2} + \frac{y}{2} + z \cdot \sin \alpha \\ 1 = \frac{y}{3} + \frac{z}{3} + x \cdot \cos \alpha \\ 1 = z + x + y \end{cases}$$

$$\text{Area} = \boxed{38.652}$$

$$\text{or } \boxed{38.653}$$

2. A regular octagon with an apothem length OM of 4.83 units is inscribed inside a circle as shown in *Figure 2*. Compute the area of quadrilateral $ABCF$.

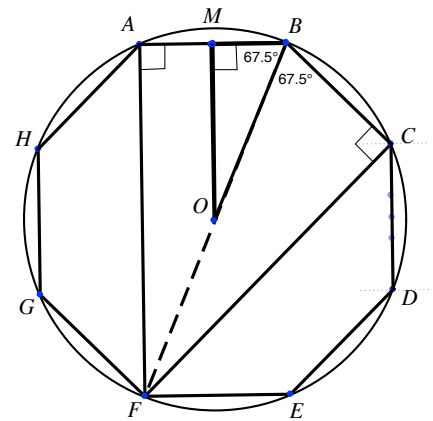


Figure 2

$$\boxed{\frac{1}{2}\sqrt{2-\sqrt{3}}}$$

3. Determine exactly the value of $\frac{2 \tan \frac{\pi}{24}}{1 + \tan^2 \frac{\pi}{24}}$.

$$\text{or } \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

$$\boxed{1006}$$

4. How many integers are in the domain of $y = \log_x \log_x \log_x (2016 - x)$?

$$\boxed{10}$$

5. Determine exactly the area of the region defined by $|x+1| < 2$ and $|y| < |x|$.

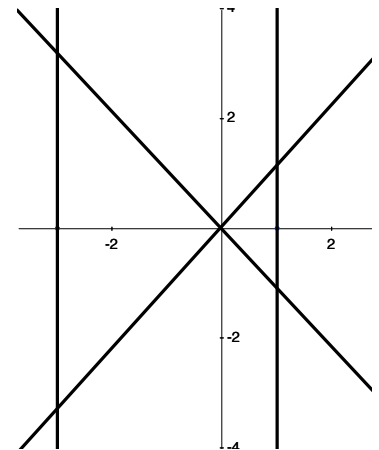
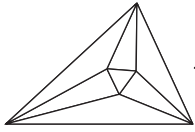


Figure 5

$$\boxed{1-\sqrt{2}}$$

6. If $\sin 2x = 2(\sin x + \cos x)$, determine exactly all values of $(\sin x)(\cos x)$.



Minnesota State High School Mathematics League

2015-16 Meet 3, Team Event

SOLUTIONS (page 2)

1. The system of equations will have no solutions when the determinant of the coefficient matrix is 0. The determinant is $\frac{1}{2}\left(\frac{1}{3} \cdot 1 - 1 \cdot \frac{1}{3}\right) - \frac{1}{2}\left(\cos \alpha - \frac{1}{3}\right) + \sin \alpha \left(\cos \alpha - \frac{1}{3}\right) = \sin \alpha \cos \alpha - \frac{\sin \alpha}{3} - \frac{\cos \alpha}{2} + \frac{1}{6} = \left(\cos \alpha - \frac{1}{3}\right)\left(\sin \alpha - \frac{1}{2}\right)$. So we need

$$\cos \alpha = \frac{1}{3} \text{ or } \sin \alpha = \frac{1}{2}. \text{ We have two choices } \sin \alpha = \frac{1}{2} \Rightarrow \cos \alpha = \pm \frac{\sqrt{3}}{2} \Rightarrow \tan \alpha = \pm \frac{\sqrt{3}}{3} \text{ or}$$

$$\cos \alpha = \frac{1}{3} \Rightarrow \sin \alpha = \pm \frac{2\sqrt{2}}{3} \Rightarrow \tan \alpha = \pm 2\sqrt{2}. \text{ The largest value of } \tan \alpha = \boxed{2\sqrt{2}}.$$

2. See Figure 2. $\tan 67.5^\circ = \frac{OM}{MB} = \frac{4.83}{MB} \Rightarrow MB = \frac{4.83}{\tan 67.5^\circ}$. AB is twice MB so $AB = \frac{9.66}{\tan 67.5^\circ}$. $\triangle MOB$ and $\triangle AFB$ are similar

with so $AF = 9.66$. The area of quadrilateral $ABCF$ is twice the area of $\triangle AFB$. So, the area of $ABCF$ is

$$2 \cdot \frac{1}{2} \cdot 9.66 \cdot \frac{9.66}{\tan 67.5^\circ} \approx \boxed{38.653}.$$

3. Using a pythagorean identity and rearranging gives $\frac{2 \tan \frac{\pi}{24}}{1 + \tan^2 \frac{\pi}{24}} = \frac{2 \tan \frac{\pi}{24}}{\sec^2 \frac{\pi}{24}} = \frac{2 \sin \frac{\pi}{24} \cos^2 \frac{\pi}{24}}{\cos \frac{\pi}{24}} = 2 \sin \frac{\pi}{24} \cos \frac{\pi}{24}$. Applying the

double angle formula for sine gives us $2 \sin \frac{\pi}{24} \cos \frac{\pi}{24} = \sin \frac{\pi}{12}$. Using the half-angle identity for sine,

$$\sin \frac{\pi}{12} = \sqrt{\frac{1 - \cos \frac{\pi}{6}}{2}} = \frac{1}{2} \sqrt{2 - \sqrt{3}}. \text{ Or, using the summation formula for sine, } \sin \frac{\pi}{12} = \sin \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

4. A base of 1 is not possible, so $x \geq 2$. Start peeling away the layers of logs. Notice we want $\log_x \log_x (2016 - x) > 0$. But in order for this to be true, $\log_x (2016 - x) > 1 \Rightarrow 2016 - x > x \Rightarrow x < 1008$. The integers that work are 2, 3, 4, ..., 1007, so there are $\boxed{1006}$ integers in all.

5. $|x + 1| < 2$ defines a region that is between the lines $x = -3$ and $x = 1$ since these are x -values that when added to 1 have an absolute value less than 2. $|y| < |x|$ defines a region that is between the lines $y = x$ and $y = -x$. The overlapping region forms two isosceles right triangles as shown in Figure 5. The big triangle has a base of 6 and a height of 3. The smaller triangle has a base of 2 and a height of 1. Therefore, the total area of the region is $\frac{1}{2} \cdot 6 \cdot 3 + \frac{1}{2} \cdot 2 \cdot 1 = \boxed{10}$.

6. Applying the double-angle identity to $\sin 2x$ and dividing through by 2 we get $\sin x \cos x = \sin x + \cos x$. Squaring both sides of the equation gives $\sin^2 x \cos^2 x = \sin^2 x + 2 \sin x \cos x + \cos^2 x \Rightarrow \sin^2 x \cos^2 x = 1 + 2 \sin x \cos x$. Rearranging the terms we find that $\sin^2 x \cos^2 x - 2 \sin x \cos x - 1 = 0 \Rightarrow (\sin x \cos x)^2 - 2(\sin x \cos x) - 1 = 0$. Solving for $\sin x \cos x$ using the quadratic

formula gives $\sin x \cos x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = 1 \pm \sqrt{2}$. However, $\sin x \cos x$ can never be larger than 1 so $1 + \sqrt{2}$ is too large.

$$\text{Thus, } \sin x \cos x = \boxed{1 - \sqrt{2}}.$$