Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

$x =$ ____________

1. Determine exactly the value of x for which $\frac{3x-5}{2}=\frac{5}{4}$.

$b =$ ____________

2. a, b, and c are positive integers such that the sum of a and b is 3 more than the value of c, the value of a is double that of b, and the value of b is five less than c, determine exactly the value of b.

____

3. In the equation $\frac{a}{b}x-\frac{b}{a}x=a+b$, if a and b are positive integers, express x in terms of a and b.

$m =$ ____________

$c =$ ____________

4. *Figure 4* shows the solution set of an inequality which can be written in the form $7 < mx + c \leq 21$, where $m > 0$. Determine exactly the values of m and c.

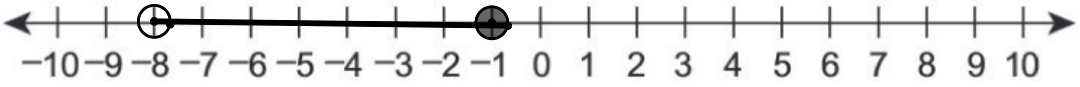


Figure 4

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

$x = \boxed{\frac{5}{2}}$

or $\boxed{2\frac{1}{2}}$

1. Determine exactly the value of x for which $\frac{3x-5}{2}=\frac{5}{4}$.

Multiply both sides by 8 to obtain 4(3x – 5) = 10 $\Rightarrow$ *12x - 20 = 10* $\Rightarrow$ *12x = 30* $\Rightarrow$ $x=\frac{30}{12}=\frac{5}{2}$.

$b = \boxed{4}$

2. a, b, and c are positive integers such that the sum of a and b is 3 more than the value of c, the value of a is double that of b, and the value of b is five less than c, determine exactly the value of b.

If the sum of a and b is 3 more than c, then a + b = c + 3. Since a is double b and b is five less than c, then a = 2b and b = c – 5 respectively. Substituting the second and third equations into the first gives us $2(c-5)+(c-5)=c+3 \Rightarrow 3(c-5)=c+3 \Rightarrow 2c=18 \Rightarrow c=9$.

Therefore, b = 9 – 5 = 4.

$x = \boxed{\frac{ab}{a-b}}$

3. In the equation $\frac{a}{b}x-\frac{b}{a}x=a+b$, if a and b are positive integers, express x in terms of a and b.

Clear the denominators by multiplying both sides of the equation by ab to obtain

$$\frac{a}{b}x-\frac{b}{a}x=a+b \Rightarrow a^2x-b^2x=ab(a+b) \Rightarrow (a^2-b^2)x=ab(a+b) \Rightarrow x=\frac{ab\cancel{(a+b)}}{(a-b)\cancel{(a+b)}} \Rightarrow x=\frac{ab}{a-b}.$$

$m = \boxed{2}$

$c = \boxed{23}$

Graders: award 1 point per correct value.

4. *Figure 4* shows the solution set of an inequality which can be written in the form $7 < mx + c \le 21$, where $m > 0$. Determine exactly the values of m and c.

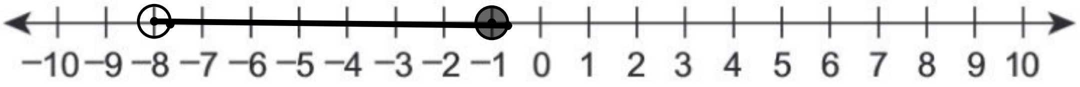


Figure 4

Figure 4 shows the solution to the inequality must be of the form $-8 < x \le -1$. *Multiplying each part of the inequality by m we obtain* $-8m < mx \le -1m$. *Adding c to each part of the inequality gives us* $-8m+c < mx+c \le -m+c$. *Therefore, –8m + c = 7 and –1m + c = 21. Solve for c in the second equation to get c = m + 21. Substituting that c value into the first equation gives –8m + m + 21 = 7* $\Rightarrow$ *–7m + 21 = 7* $\Rightarrow$ *m = 2. Therefore, c = 23.*

Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

________________ 1. Determine exactly the area of a triangle whose sides have lengths 8, 15, 17.

$b =$ ________________ 2. Two triangles share a base of length b, but one triangle's height is 3 more than the other's. If the difference of their areas is 6, determine the exact value of b.

$BC =$ ________________ 3. In *Figure* 3, $\triangle ABC$, D lies on $\overline{BC}$, E lies on $\overline{AC}$, and F lies on $\overline{AB}$ such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are bisectors of angles BAC, ABC, and BCA, respectively. If $CE = 9$, $EA = 6$, and $AF = 4$, determine BC exactly.

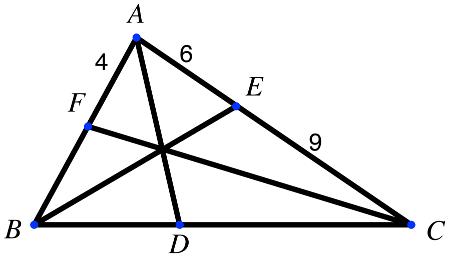


Figure 3

________________ 4. If the lengths of a triangle's sides are 10, 12, and 14, determine exactly the length of the median that intersects the side of length 12.

Name: ____________________________ Team: ____________________________

Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event B

SOLUTIONS

$\boxed{60}$ 1. Determine exactly the area of a triangle whose sides have lengths 8, 15, 17.

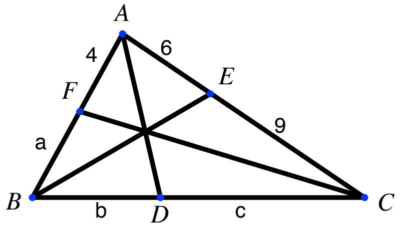
The sides form a Pythagorean triple so the area is $\frac{1}{2}bh = \frac{1}{2}\cdot 8\cdot 15 = 60.$

$b = \boxed{4}$ 2. Two triangles share a base of length b, but one triangle's height is 3 more than the other's. If the difference of their areas is 6, determine the exact value of b.

Let x and x + 3 be the heights of the triangles respectively. The difference of the areas can be written as $\frac{1}{2}(x+3)b - \frac{1}{2}(x)b = 6 \Rightarrow bx + 3b - bx = 12 \Rightarrow b = 4.$

$BC = \boxed{10}$ 3. In $\triangle ABC$, D lies on $\overline{BC}$, E lies on $\overline{AC}$, and F lies on $\overline{AB}$ such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are bisectors of angles BAC, ABC, and BCA, respectively. If $CE = 9$, $EA = 6$, and $AF = 4$, determine BC exactly.

Figure 3



Let BF = a, BD = b, and DC = c. By the Angle Bisector Theorem, $\frac{AB}{BC} = \frac{AE}{EC} \Rightarrow \frac{a+4}{b+c} = \frac{2}{3} \Rightarrow 2b + 2c = 3a + 12.$

Applying the Angle Bisector Theorem again gives $\frac{AC}{BC} = \frac{AF}{FB} \Rightarrow \frac{15}{b+c} = \frac{4}{a} \Rightarrow 15a = 4b + 4c \Rightarrow 3a = \frac{4b+4c}{5}.$

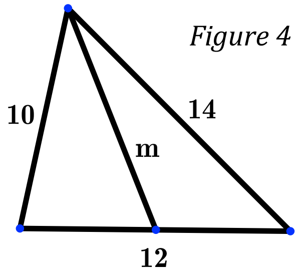
Substituting into the first equation gives $2b + 2c = \frac{4b}{5} + \frac{4c}{5} + 12 \Rightarrow \frac{6(b+c)}{5} = 12 \Rightarrow b + c = \frac{5}{6}\cdot 12 = 10.$

$\boxed{4\sqrt{7}}$ 4. If the lengths of a triangle's sides are 10, 12, and 14, determine exactly the length of the median that intersects the side of length 12.

See Figure 4. Using Stewart's Theorem we have

$$10^2\cdot 6 + 14^2\cdot 6 = 12\left(m^2 + 6\cdot 6\right) \Rightarrow \frac{100+196}{2} = m^2 + 36 \Rightarrow m^2 = 148 - 36 = 112 \Rightarrow m = \sqrt{112} = 4\sqrt{7}.$$

Figure 4



Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

tan θ = __________

1. If $0<\theta<\frac{\pi}{2}$ and $\sin\theta=\frac{3}{7}$, determine exactly the value of tan θ.

cos θ = __________

2. If $\pi<\theta<\frac{3\pi}{2}$ and $\cos 2\theta=\frac{1}{8}$, determine exactly the value of cos θ.

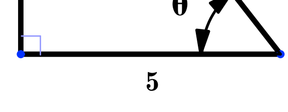
____

3. Determine exactly the value of x if $\sin x\cos 17° = -\cos x \sin 17°$, where $0° < x < 180°$.

sin 3θ = __________

4. Given the angle θ as defined by *Figure* 4, determine exactly the value of sin 3θ.

Figure 4



Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\tan\theta = \boxed{\dfrac{3\sqrt{10}}{20}}$

1. If $0<\theta<\dfrac{\pi}{2}$ and $\sin\theta=\dfrac{3}{7}$, determine exactly the value of tan *θ.*

See Figure 1. $\tan\theta=\dfrac{3}{\sqrt{40}}=\dfrac{3\sqrt{40}}{40}=\dfrac{6\sqrt{10}}{40}=\dfrac{3\sqrt{10}}{20}.$

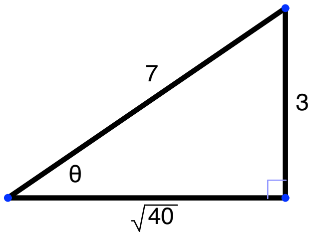


Figure 1

$\cos\theta = \boxed{\dfrac{-3}{4}}$

2. If $\pi<\theta<\dfrac{3\pi}{2}$ and $\cos 2\theta=\dfrac{1}{8}$, determine exactly the value of cos *θ.*

$\cos 2\theta=2\cos^2\theta-1=\dfrac{1}{8} \Rightarrow \cos^2\theta=\dfrac{9}{16} \Rightarrow \cos\theta=\pm\dfrac{3}{4}$. *However,* $\pi<\theta<\dfrac{3\pi}{2}$, *so* $\cos\theta=\dfrac{-3}{4}$.

$x = \boxed{163°}$

3. Determine exactly the value of *x* if $\sin x\cos 17°=-\cos x\sin 17°$, where 0° < *x* < 180°.

Rewrite the equation to get $\sin x\cos 17°+\cos x\sin 17°=0$. *Using the sum identity for sine,* $\sin x\cos 17°+\cos x\sin 17°=\sin\left(x+17°\right)=0$. *Sine is 0 at 180°. So x + 17 ° = 180 ° , and x = 163°.*

$\sin 3\theta = \boxed{\dfrac{-828}{2197}}$

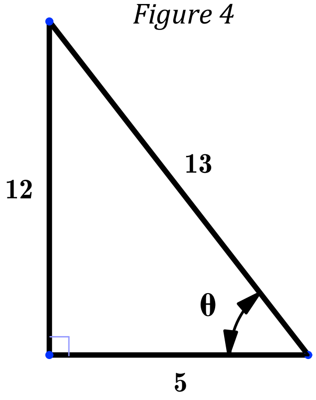
4. Given the angle *θ* as defined by *Figure* 4, determine exactly the value of sin 3*θ*.

Using the sum identity for sine we obtain $\sin 3\theta=\sin\left(2\theta+\theta\right)=\sin 2\theta\cdot\cos\theta+\cos 2\theta\cdot\sin\theta$.

Applying the double angle formulas for sine and cosine we get
$\sin 2\theta\cdot\cos\theta+\cos 2\theta\cdot\sin\theta=2\sin\theta\cos^2\theta+\left(2\cos^2\theta-1\right)\sin\theta$.

So, $\sin 3\theta=2\sin\theta\cos^2\theta+\left(2\cos^2\theta-1\right)\sin\theta=2\left(\dfrac{12}{13}\right)\left(\dfrac{5}{13}\right)^2+\left(2\left(\dfrac{5}{13}\right)^2-1\right)\left(\dfrac{12}{13}\right)=\dfrac{-828}{2197}$.

Figure 4



Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

________________ 1. What is the <u>sum</u> of the x-coordinates of the x-intercepts of $(x-2)^2+(y-3)^2=10$?

________________ 2. Lines ℓ_1 and ℓ_2 intersect at (3, 7). If the sum of the lines' slopes is 5, and ℓ_1 passes through (9,1), determine exactly the y-intercept of ℓ_2.

$(x, y) =$ __________ 3. Line ℓ passes through $P = (5, 10)$. If point Q is the point on ℓ closest to the origin O and $\overline{OQ}$ has a slope of $\frac{1}{2}$, determine exactly the coordinates (x, y) of Q.

$c^2 =$ __________ 4. Determine exactly the value of c^2 so that the circle $x^2+y^2=c^2$ is tangent to the line $cx + 2y = 6$.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2015-16 Meet 2, Individual Event D

SOLUTIONS

sum = $\boxed{4}$

1. What is the sum of the x-coordinates of the x-intercepts of $(x-2)^2+(y-3)^2=10$?

Set y = 0 to get $(x-2)^2+9=10 \Rightarrow x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$. *So, the x-coordinates of the x-intercepts are 3 and 1 and their sum is 4.*

$\boxed{(0,-11)}$

Graders: Also accept −11

2. Lines ℓ_1 and ℓ_2 intersect at (3,7). If the sum of the lines' slopes is 5, and ℓ_1 passes through (9,1), determine exactly the y-intercept of ℓ_2.

Line ℓ_1 goes through points (3, 7) and (9, 1). ℓ_1 then has a slope of $\frac{1-7}{9-3}=-1$. *Line ℓ_2 then has a slope of 6 since sum of the lines' slopes is 5. Using point slope on ℓ_2, we obtain $y - 7 = 6(x - 3) \Rightarrow y = 6x - 11$, which has a y-intercept of (0, −11).*

(x, y) = $\boxed{(8,4)}$

Graders: Award 1 point per correct coordinate.

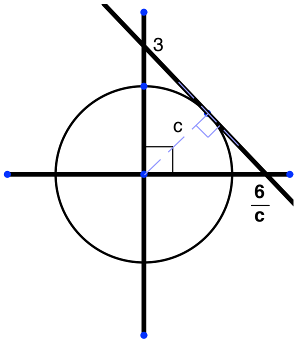
3. Line ℓ passes through P = (5, 10). If point Q is the point on ℓ closest to the origin O and $\overline{OQ}$ has a slope of $\frac{1}{2}$, determine exactly the coordinates (x, y) of Q.

Since $\overline{OQ}$ has a slope of $\frac{1}{2}$, ℓ has a slope of −2. The equation of the line through $\overline{OQ}$ is $y=\frac{1}{2}x$ The equation of ℓ is $y - 10 = -2(x - 5) \Rightarrow$ $y = -2x + 20$. The point of intersection of these lines, which is the point closest to the origin, occurs when $-2x+20=\frac{1}{2}x \Rightarrow x=8$. So, y = 4 and the coordinates of Q are (8, 4).

$\boxed{2\sqrt{10}-2}$

4. Determine exactly the value of c^2 so that the circle $x^2+y^2=c^2$ is tangent to the line $cx + 2y = 6$.

Figure 4



Since c is the radius of the circle, c > 0. The line cx + 2y = 6 has x and y-intercepts of $\frac{6}{c}$ and 3 and creates a right triangle in the first quadrant with hypotenuse $\sqrt{9+\frac{36}{c^2}}$. The area of the triangle can be represented as $\frac{1}{2}\cdot 3\cdot\frac{6}{c}=\frac{9}{c}$ or $\frac{1}{2}\cdot c\cdot\sqrt{9+\frac{36}{c^2}}$.

Equating these areas gives

$$\frac{1}{2}c\sqrt{9+\frac{36}{c^2}}=\frac{9}{c} \Rightarrow \frac{18}{c^2}=\sqrt{9+\frac{36}{c^2}} \Rightarrow \frac{18^2}{c^4}=9+\frac{36}{c^2} \Rightarrow c^4+4c^2-36=0.$$

Solving this quadratic in c^2, we get c^2 = $2\sqrt{10}-2$.

Minnesota State High School Mathematics League

2015-16 Meet 2, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

1. Lines $\ell_1, \ell_2, and \ell_3$ are all parallel. ℓ_1 is the line $3x - 7y = 11$, ℓ_2 passes through (20, 15), and ℓ_3 passes through (100, c). Determine exactly the <u>least</u> integer c for which ℓ_3 is between the two other lines.

a =

b =

2. In $\triangle ABC, AC = BC, AB = 10$, and $\angle C$ is acute. The range of values that the length of the altitude from A to $\overline{BC}$ can take on is represented by the open interval (a, b). Determine the exact values of a and b.

3. Determine the equation of the line that passes through the y-intercept of $y = x^2 + 4x - 5$ and the center of the circle passing through the x-intercepts and vertex of $y = x^2 + 4x - 5$.

4. Given the angle θ as defined by *Figure 4*, determine exactly the value of cos 4θ.

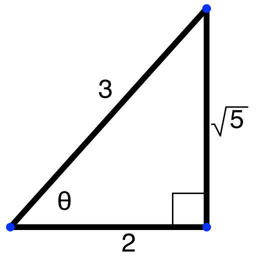


Figure 4

r =

5. Determine exactly the length of the <u>longest</u> radius r such that three solid non-overlapping circles of radius r can simultaneously fit inside the region bounded by $|x| \le 20$ and $|y| \le 15$.

6. The edges of a regular tetrahedron (a four-faced pyramid with equilateral triangles as faces) have a length of 6. The angle bisectors of one face intersect at P and the angle bisectors of a second face intersect at Q. Determine PQ exactly.

Team: ________________________

Minnesota State High School Mathematics League

2015-16 Meet 2, Team Event

SOLUTIONS (page 1)

$\boxed{42}$

1. Lines $\ell_1, \ell_2, and\, \ell_3$ are all parallel. ℓ_1 is the line $3x - 7y = 11$, ℓ_2 passes through (20, 15), and ℓ_3 passes through (100, c). Determine exactly the least integer c for which ℓ_3 is between the two other lines.

$a = \boxed{5\sqrt{2}}$

$b = \boxed{10}$

Graders: award 2 points per correct value.

2. In $\triangle ABC$, $AC = BC$, $AB = 10$, and $\angle C$ is acute. The range of values that the length of the altitude from A to $\overline{BC}$ can take on is represented by the open interval (a, b). Determine the exact values of a and b.

$\boxed{y = \frac{-1}{2}x - 5}$

3. Determine the equation of the line that passes through the y-intercept of $y = x^2 + 4x - 5$ and the center of the circle passing through the x-intercepts and vertex of $y = x^2 + 4x - 5$.

$\boxed{\frac{-79}{81}}$

4. Given the angle θ as defined by *Figure 4*, determine exactly the value of cos 4θ.

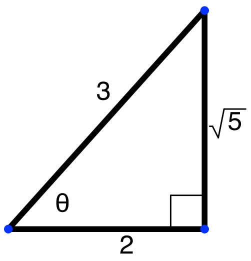


Figure 4

$r = \boxed{80 - 10\sqrt{51}}$

5. Determine exactly the length of the longest radius r such that three solid non-overlapping circles of radius r can simultaneously fit inside the region bounded by $|x| \le 20$ and $|y| \le 15$.

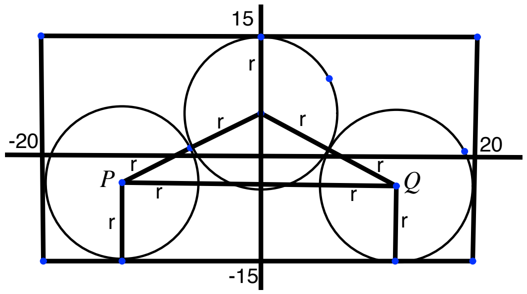


Figure 5

$\boxed{2}$

6. The edges of a regular tetrahedron (a four-faced pyramid with equilateral triangles as faces) have a length of 6. The angle bisectors of one face intersect at P and the angle bisectors of a second face intersect at Q. Determine PQ exactly.

Minnesota State High School Mathematics League

2015-16 Meet 2, Team Event

SOLUTIONS (page 2)

1. *ℓ_1, ℓ_2, and ℓ_3 are all parallel and therefore have the same slope of $\frac{3}{7}$. ℓ_2 is a line with the equation 3x – 7y = –45 since it has a slope of $\frac{3}{7}$ and it passes through (20, 15). ℓ_3 is a line of the form 3x – 7y = 300 – 7c. Line ℓ_3 is between the other two lines, so* $-45<300-7c<11 \Rightarrow -345<-7c<-289 \Rightarrow \frac{345}{7}>c>\frac{289}{7} \Rightarrow 49\frac{2}{7}<c<41\frac{2}{7} \Rightarrow c=\boxed{42}$.

2. *Let h be the length of the altitude. If* $m\angle C=90°$, *then* $\triangle ABC$ *is an isosceles right triangle and leg* $\overline{AC}$ *would be the altitude with length* $5\sqrt{2}$, *and thus* $h>5\sqrt{2}$. *As the legs of the triangle increase,* $\angle C$ *decreases and the altitude hits* $\overline{BC}$ *at a point closer to B. This means the altitude is approaching the length of* $\overline{AB}$, *and thus* $h<10$. *Therefore, a is* $\boxed{5\sqrt{2}}$ *and b is* $\boxed{10}$.

3. *The equation of our line must be of the form y = mx – 5 since it runs through the y-intercept of* $y=x^2+4x-5$. *The center of our circle lies on the axis of symmetry of our parabola and therefore is of the form (–2, k). This center is 9 + k units away from the vertex (–2, –9) and* $\sqrt{(-2-1)^2+(k-0)^2}=\sqrt{9+k^2}$ *away from the x - intercept of (1, 0). Equating these two distances, we get* $9+k=\sqrt{9+k^2} \Rightarrow (9+k)^2=9+k^2 \Rightarrow 81+18k+k^2=9+k^2 \Rightarrow k=-4$. *The slope between (–2, –4) and (0, –5) is* $\frac{-1}{2}$, *which means the equation of our line is* $\boxed{y=\frac{-1}{2}x-5}$.

4. *First, apply the double-angle identity for cosine:* $\cos 4\theta=\cos^2 2\theta-\sin^2 2\theta$. *Then, applying the double angle formulas again to sine and cosine,* $\cos^2 2\theta-\sin^2 2\theta=\left(1-2\sin^2\theta\right)^2-\left(2\sin\theta\cos\theta\right)^2$. *Substituting in sine and cosine values from Figure 4, we get* $\left(1-2\left(\frac{5}{9}\right)\right)^2-\left(2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)\right)^2=\left(\frac{-1}{9}\right)^2-\left(\frac{4\sqrt{5}}{9}\right)^2=\boxed{\frac{-79}{81}}$.

5. *Figure 5 shows the optimal placement for the non-overlapping circles is in the bottom left and bottom right corners, with the third circle touching the others and touching the top edge (or the vertically flipped arrangement). The middle circle will have its center at (0, 15 – r), the bottom right circle will have its center at (20 – r, –15 + r). Since the circles are tangent to one another, the distance between their centers is 2r. Using the Distance Formula we get:*

$$2r=\sqrt{(20-r)^2+((-15+r)-(15-r))^2} \Rightarrow 2r=\sqrt{(20-r)^2+(2r-30)^2} \Rightarrow 4r^2=5r^2-160r+1300 \Rightarrow r^2-160r+1300=0 \Rightarrow r=80\pm10\sqrt{51}.$$

However, $80+10\sqrt{51}$ *is too large for the given region, so* $r=\boxed{80-10\sqrt{51}}$.

6. *The angle bisectors of the triangles are also the medians for an equilateral triangle and so they meet at the centroid of the triangle, which divides the angle bisector into two parts whose ratio is 2:1. If we draw segments parallel to the fourth face through the centroids of the other faces, these segments form* $\triangle ABC$, *whose sides are 4, as shown in Figure 6. Points P and Q are the midpoints of the sides of* $\triangle ABC$ *so* $\overline{PQ}$ *is a midline whose length is half the length of a side, namely* $\boxed{2}$.

Figure 6