Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

________ *Turkish liras* 1. Determine exactly how many Turkish liras 1 dollar will buy if a hotel room costs $54 for 81 Turkish liras.

________ 2. Determine the exact value of $\frac{(.\bar{7})}{(.\overline{63})}$.

$\min(a+b)=$ ________ 3. Let a and b be positive integers. If the greatest common factor of a and b is 10 and the least common multiple of a and b is 60, determine the minimum possible value of $a + b$.

$\min n=$ ________ 4. Given $S=\{1, 2, 3, \ldots, n-1, n\}$, determine the <u>least</u> value of n such that there are exactly 1.25 times as many multiples of 19 in S as multiples of 23.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event A

SOLUTIONS

$\boxed{1\frac{1}{2}}$ *or* $\boxed{1.5}$ *or* $\boxed{\frac{3}{2}}$

1. Determine exactly how many Turkish liras 1 dollar will buy if a hotel room costs \$54 for 81 Turkish liras.

$$\frac{81\ Turkish\ liras}{\$54}=\frac{9\ Turkish\ liras}{\$6}=\frac{3\ Turkish\ liras}{\$2}=\frac{1\frac{1}{2}\ Turkish\ liras}{\$1}=\frac{1.5\ Turkish\ liras}{\$1}$$

$\boxed{\frac{11}{9}}$ *or* $\boxed{1\frac{2}{9}}$ *or* $\boxed{1.\overline{2}}$

2. Determine the exact value of $\frac{(.\overline{7})}{(.\overline{63})}$.

$$\frac{.\overline{7}}{.\overline{63}}=\frac{\frac{7}{9}}{\frac{63}{99}}=\frac{\frac{7}{9}}{\frac{7}{11}}=\frac{7}{9}\cdot\frac{11}{7}=\frac{11}{9}=1\frac{2}{9}.$$

$\min(a+b)=\boxed{50}$

3. Let a and b be positive integers. If the greatest common factor of a and b is 10 and the least common multiple of a and b is 60, determine the minimum possible value of a + b.

Because a and b have a greatest common factor of 10, a and b must both be divisible by 10. This means the units digit of both a and b is 0. Since the LCM(a,b) = 60, a and b can each not be any larger than 60. Thus, the only values a or b can take are 10, 20, 30, or 60. Our possible ordered pairs for (a,b) are (10,60), (60,10), (20,30) and (30,20), so a + b takes a minimum value of 50.

$\min n=\boxed{95}$

4. Given $S=\{1,2,3,\ldots,n-1,n\}$, determine the <u>least</u> value of n such that there are exactly 1.25 times as many multiples of 19 in S as multiples of 23.

The multiples of 19 in set S are 19, 38, 57, 76, 95, 114, . . . The multiples of 23 in set S are 23, 46, 69, 92, 115, . . . We seek to find the least value n, so try looking for exactly 5 multiples of 19 and 4 multiples of 23. Since 5 · 19 = 95 and 4 · 23 = 92, if n = 95, there are exactly 5 multiples of 19 and 4 multiples of 23. Thus, n = 95.

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$m\angle BCD =$ ____________

1. In *Figure 1*, if $\triangle ABC$ is equilateral and $\overrightarrow{CD}$ is parallel to $\overline{AB}$, calculate the measure of $\angle BCD$.

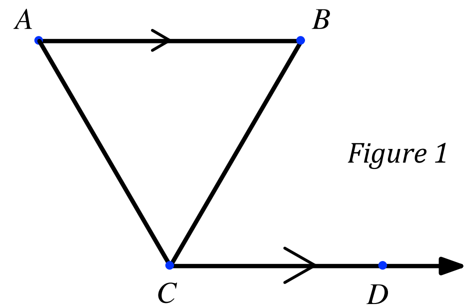


Figure 1

____________ *hours*

2. Town *A* is located exactly 120 miles north of town *B*. If Sue hops in a car and drives directly east from town *B* at 50 mph, calculate how many hours (as a decimal) it will take for Sue to be exactly 241 miles from town *A* as the crow flies.

AC = ____________

3. Right $\triangle ABC$ has $\overline{AD}$ congruent to $\overline{AB}$, as shown in *Figure 3*. If *BC* = 7 and *DC* = 1, determine exactly *AC*, the length of the hypotenuse.

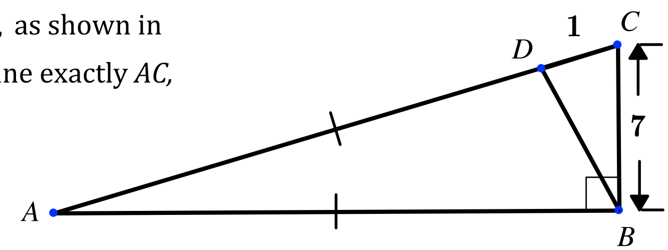


Figure 3

FE = ____________

4. Three right triangles *ABC*, *DCE*, and *FEG* are lined up in a row and mutually connected by $\overline{AG}$ to form one big right triangle *ABG* (*Figure 4*). If $\angle BAC \cong \angle CDE \cong \angle DEC \cong \angle BCA$, AG = 12 and $m\angle BGA = 30°$, determine exactly the length *FE*.

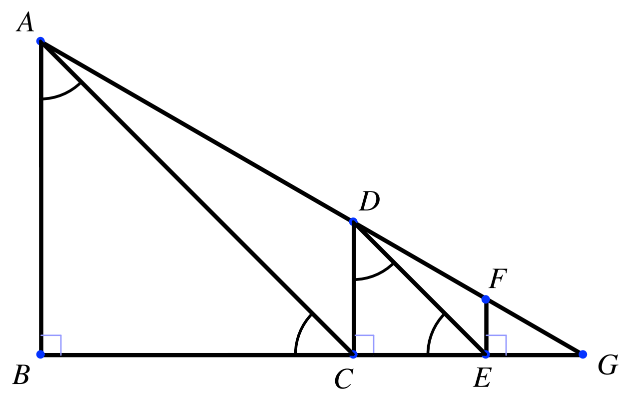


Figure 4

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event B

SOLUTIONS

$m\angle BCD = \boxed{60°}$

1. In *Figure 1*, if $\triangle ABC$ is equilateral and $\overrightarrow{CD}$ is parallel to $\overline{AB}$, calculate the measure of $\angle BCD$.

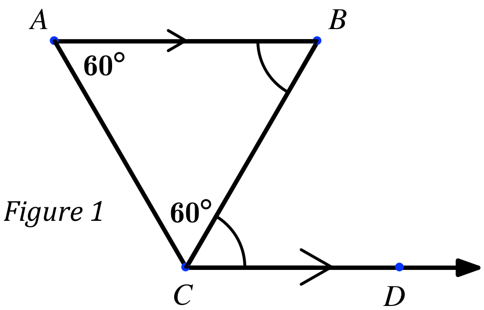


Figure 1

$m\angle ABC = m\angle BCD = 60°$ since $\angle ABC$ and $\angle BCD$ are alternate interior angles and $\triangle ABC$ is equilateral.

$\boxed{4.18}$ *hours*

2. Town *A* is located exactly 120 miles north of town *B*. If Sue hops in a car and drives directly east from town *B* at 50 mph, calculate how many hours (as a decimal) it will take for Sue to be exactly 241 miles from town *A* as the crow flies.

Let C equal Sue's location when she is 241 miles from town A. Form a right triangle between A, B, and C. Using the Pythagorean Theorem, we find $BC = \sqrt{241^2 - (120)^2} = \sqrt{43681} = 209$.

The time it takes for Sue to travel 209 miles is then $\frac{209\ mi.}{50\ mph} \approx 4.18\ hours$.

AC = $\boxed{25}$

3. Right $\triangle ABC$ has $\overline{AD}$ congruent to $\overline{AB}$, as shown in *Figure 3*. If *BC* = 7 and *DC* = 1, determine exactly *AC*, the length of the hypotenuse.

Let AD = AB = x, as shown in Figure 3. By the Pythagorean Theorem,

$x^2 + 7^2 = (1+x)^2 \Rightarrow x^2 + 49 = 1 + 2x + x^2 \Rightarrow 49 = 1 + 2x \Rightarrow x = 24$,

which means AC = 1+x = 25.

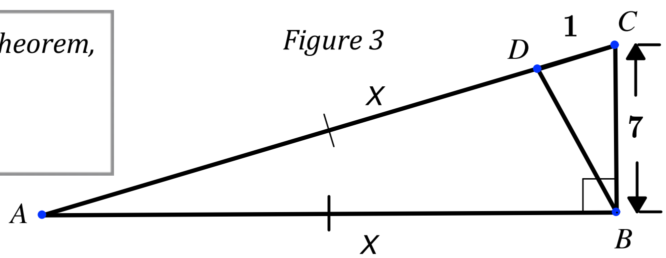


Figure 3

FE = $\boxed{8 - 4\sqrt{3}}$

4. Three right triangles *ABC*, *DCE*, and *FEG* are lined up in a row and mutually connected by $\overline{AG}$ to form one big right triangle *ABG* (*Figure 4*). If $\angle BAC \cong \angle CDE \cong \angle DEC \cong \angle BCA$, AG = 12 and $m\angle BGA = 30°$, determine exactly the length *FE*.

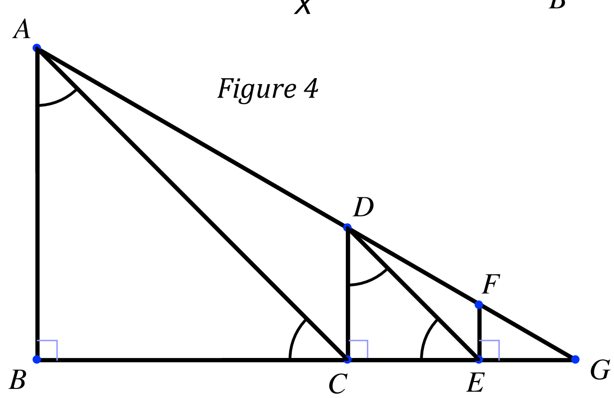


Figure 4

ABC and DCE are both 45°-45°-90° right triangles and AB = 6, so BG = $6\sqrt{3}$. *CG is* $6\sqrt{3} - 6$ *because BC = 6. Triangle DCG is a 30°-60°-90° right triangle, so DC =* $6 - 2\sqrt{3}$ *and CE =* $6 - 2\sqrt{3}$.

EG = $(6\sqrt{3} - 6) - (6 - 2\sqrt{3}) = 8\sqrt{3} - 12$. *Therefore, FE =* $\frac{8\sqrt{3} - 12}{\sqrt{3}} = 8 - 4\sqrt{3}$.

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Determine exactly the value of $\sin\frac{\pi}{3} - \cos 3\pi$.

$1+\sin^2 A=$ ________________ 2. If $\tan A = -\frac{\sqrt{39}}{5}$ and $\cos A = \frac{5}{8}$, determine exactly the value of $1+\sin^2 A$.

________________ 3. Determine exactly the <u>smallest</u> angle x > 0 (in radians) for which the graphs $y = 6\cos(7x+\pi)$ and $y = 3$ intersect .

________________ 4. Determine the <u>number</u> of angles A, where A is an integer $0° < A < 180°$, for which $\cot A < \tan 111°$.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\frac{\sqrt{3}}{2}+1}$ ***or*** $\boxed{\frac{\sqrt{3}+2}{2}}$

1. Determine exactly the value of $\sin\frac{\pi}{3}-\cos 3\pi$.

$$=\frac{\sqrt{3}}{2}--1=\frac{\sqrt{3}}{2}+1=\frac{\sqrt{3}+2}{2}.$$

$\tan x = \boxed{\frac{103}{64}}$ ***or*** $\boxed{1\frac{39}{64}}$

2. If $\tan A=-\frac{\sqrt{39}}{5}$ and $\cos A=\frac{5}{8}$, determine exactly the value of $1+\sin^2 A$.

$$\tan A=\frac{\sin A}{\cos A}\Rightarrow \sin A=\tan A\bullet\cos A=-\frac{\sqrt{39}}{5}\bullet\frac{5}{8}=-\frac{\sqrt{39}}{8}.$$

Therefore, $1+\sin^2 A=1+\left(-\frac{\sqrt{39}}{8}\right)^2=\frac{103}{64}=1\frac{39}{64}.$

$\boxed{\frac{2\pi}{21}}$

3. Determine exactly the smallest angle x > 0 (in radians) for which the graphs $y=6\cos(7x+\pi)$ and $y = 3$ intersect .

The graphs intersect when $6\cos(7x+\pi)=3 \Rightarrow \cos(7x+\pi)=\frac{1}{2}$.
The cosine of an angle is .5 when the angle is equal to $\frac{\pi}{3}+2k\pi$ *or* $\frac{5\pi}{3}+2k\pi$, *where k is an integer value, as shown in figure 3. The smallest positive angle occurs when* $7x+\pi=\frac{5\pi}{3}\Rightarrow x=\frac{2\pi}{21}$.

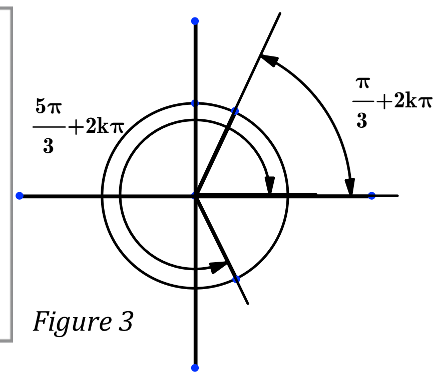


Figure 3

$\boxed{20}$

4. Determine the number of angles A, where A is an integer $0° < A < 180°$, for which $\cot A<\tan 111°$.

111° is an angle in the second quadrant and since tangent values are negative in that quadrant, we can eliminate all angle values A in the first quadrant whose cotangents are positive. It is also important to note that 111° is an angle 21° from the positive y-axis. So $\tan 111°$ *is the same as* $\cot(180°-21°)=\cot 159°$, *see figure 4. As we choose angles closer to 180°, the cotangent values become more negative. Thus the angles for which* $\cot A<\tan 111°$ *are* $160°, 161°, \ldots 179°$. *There are 20 such angles.*

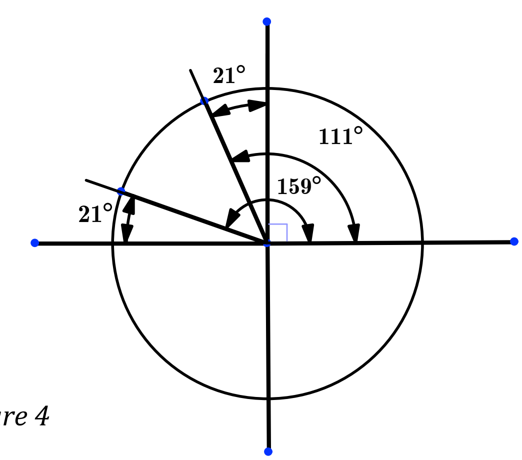


Figure 4

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

$x =$ ________________ 1. Let $f(x) = x + 3$ and $g(x) = x^2$. Determine exactly the value(s) of x for which $g(f(x)) = 0$.

________________ 2. Find the remainder when $2x^3 - 9x^2 + 14x - 6$ is divided by $x + 2$.

$k =$ ________________ 3. The quadratic function $f(x) = 3x^2 + kx + 5$, where k is an integer, does not have any real roots. What is the greatest possible value of k ?

$b =$ ________________ 4. Let $f(x) = 2x^2 - bx + 7$ and $g(x) = 2(x - c)^2 - 43$.
If $f(x)$ and $g(x)$ are equal for all values of x, determine exactly all possible values of b.

Name: ______________________________ Team: ______________________________

Minnesota State High School Mathematics League

2015-16 Meet 1, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{-3}$ 1. Let $f(x)=x+3$ and $g(x)=x^2$. Determine exactly the value(s) of x for which $g(f(x))=0$.

$g(f(x))=(x+3)^2=0 \Rightarrow x=-3$

$\boxed{-86}$ 2. Find the remainder when $2x^3-9x^2+14x-6$ is divided by $x+2$.

Using synthetic division we get

−2	2	−9	14	−6
	2	−13	40	$\boxed{-86}$

$\Rightarrow$ *remainder* $=-86$. *Another way to obtain the remainder is by using the Remainder Theorem to get* $2(-2)^3-9(-2)^2+14(-2)-6=-86$.

$k = \boxed{7}$ 3. The quadratic function $f(x)=3x^2+kx+5$, where k is an integer, does not have any real roots. What is the greatest possible value of k?

A quadratic function does not have any real roots when the value of its discriminant is less than 0. Therefore, $k^2-4\bullet 3\bullet 5<0 \Rightarrow k^2<60 \Rightarrow -\sqrt{60}<k<\sqrt{60}$. *Since k is the largest integer value for which this is true, k = 7.*

$b = \boxed{20}$ or $b = \boxed{-20}$ 4. Let $f(x)=2x^2-bx+7$ and $g(x)=2(x-c)^2-43$. If $f(x)$ and $g(x)$ are equal for all values of x, determine exactly all possible values of b.

Graders: Award only 1 point if any extra solutions are given, or for only one correct solution.

Rewrite g(x) in standard form to get $g(x)=2x^2-4cx+2c^2-43$. *Equating the coefficients of each function we obtain* $b=4c$ *and* $2c^2-43=7$. *Solving the second equation gives us c values of −5 and 5. This means b = 20 or b = −20.*

Minnesota State High School Mathematics League

2015-16 Meet 1, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

_______________ 1. If $.\overline{AB}+.\overline{CD}=\frac{1}{3}$ with $0<.\overline{AB}<.\overline{CD}$, determine exactly the <u>sum</u> of all pairs $(.\overline{AB}\ ,\ .\overline{CD})$ satisfying those conditions.

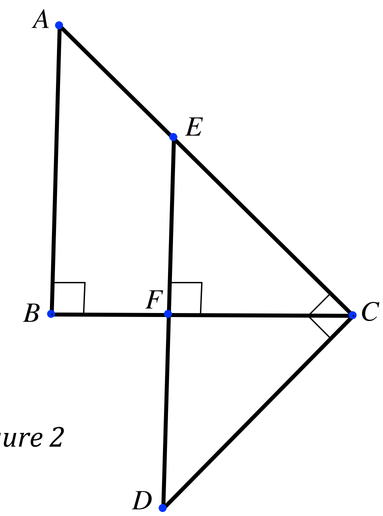


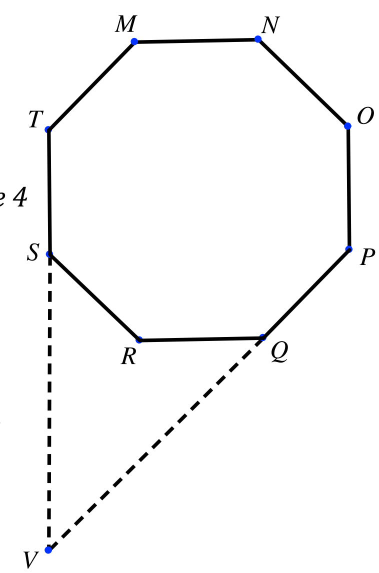
Figure 2

EF = _______________ 2. In *Figure 2*, overlapping right triangles *ABC* and *CDE* are drawn such that *AC* = *CD* and $\overline{ED}$ is perpendicular to $\overline{BC}$. If *AB* = *x* and $\angle BAC=\alpha$, express the length *EF* in terms of *x* and α only.

g(*x*) = _______________ 3. Let *g*(*x*) be a linear function and let $f\left(x\right)=x^2-6x+c$, where the roots of f(x) are $3\pm4i$. If the solutions to the equation $f\left(x\right)=g\left(x\right)$ are *x* = 0 and the x coordinate of the axis of symmetry of *f*(*x*), write the formula for *g*(*x*).

TV = _______________ 4. A regular octagon *MNOPQRST* has sides of length 5, as shown in *Figure 4*. If sides $\overline{PQ}$ and $\overline{ST}$ are extended until they meet at point V, find the length of $\overline{TV}$ exactly.

Figure 4



k= _______________ 5. Determine exactly the value of *k*, where k is positive, for which the polynomial $17x^2-19x+k$ has roots that are the sines of acute angles of some right triangle *ABC*.

_______________ 6. Let $N=p^{2017}-4p^{2016}+4p^{2015}$, where N is a positive number. If *p* is a prime number, determine the <u>least</u> <u>possible</u> number of factors of N.

Team: ______________________________

Minnesota State High School Mathematics League

2015-16 Meet 1, Team Event

SOLUTIONS (page 1)

$\boxed{\frac{16}{3}}$ **or** $\boxed{5\frac{1}{3}}$

1. If $.\overline{AB}+.\overline{CD}=\frac{1}{3}$ with $0<.\overline{AB}<.\overline{CD}$, determine exactly the <u>sum</u> of all pairs $(.\overline{AB}\ ,\ .\overline{CD}\ )$ satisfying those conditions.

$EF=\boxed{\frac{x}{\tan\alpha}}$ **or** $\boxed{x\cot\alpha}$

2. In *Figure 2*, overlapping right triangles *ABC* and *CDE* are drawn such that *AC* = *CD* and $\overline{ED}$ is perpendicular to $\overline{BC}$. If *AB* = *x* and $\angle BAC=\alpha$, express the length *EF* in terms of *x* and α only.

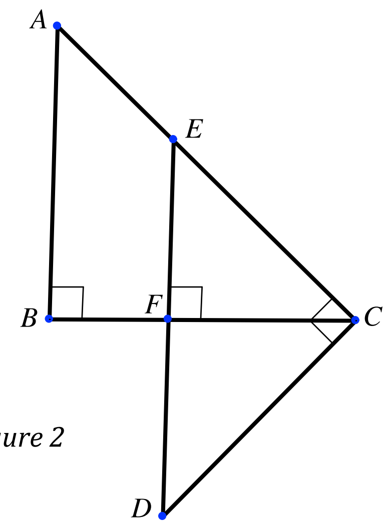


Figure 2

$g(x)=\boxed{-3x+25}$

3. Let $g(x)$ be a linear function and let $f(x)=x^2-6x+c$, where the roots of f(x) are $3\pm 4i$. If the solutions to the equation $f(x)=g(x)$ are $x=0$ and the x coordinate of the axis of symmetry of $f(x)$, write the formula for $g(x)$.

$TV=\boxed{10+5\sqrt{2}}$

4. A regular octagon *MNOPQRST* has sides of length 5, as shown in *Figure 4*. If sides $\overline{PQ}$ and $\overline{ST}$ are extended until they meet at point V, find the length of $\overline{TV}$ exactly.

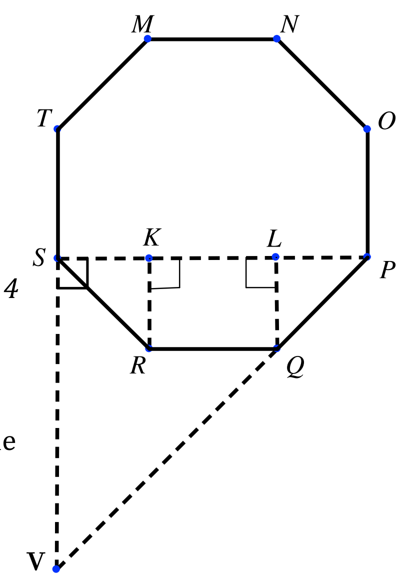


Figure 4

$k=\boxed{\frac{36}{17}}$ **or** $\boxed{2\frac{2}{17}}$

5. Determine exactly the value of *k*, where k is positive, for which the polynomial $17x^2-19x+k$ has roots that are the sines of acute angles of some right triangle *ABC*.

$\boxed{2016}$

6. Let $N=p^{2017}-4p^{2016}+4p^{2015}$, where N is a positive number. If p is a prime number, determine the <u>least</u> <u>possible</u> number of factors of N.

Minnesota State High School Mathematics League

2015-16 Meet 1, Team Event

SOLUTIONS (page 2)

1. *If N =* $.\overline{AB}$, *then 100N =* $AB.\overline{AB}$ *and subtraction by N gives 99N = 10A + B. So,* $.\overline{AB}=\frac{10A+B}{99}$ *and, similarly,* $.\overline{CD}=\frac{10C+D}{99}$. *Thus,* $\frac{10(A+C)+(B+D)}{99}=\frac{1}{3}=\frac{33}{99}$ *so 10(A +C) +(B +D) = 33. There are two possible cases that apply. The first is A + C = 3 and B + D = 3, then A = 0 and C = 3, in which (B,D) = (1,2), (2,1), and (3,0), or A = 1 and C = 2, in which (B,D) = (0,3), (1,2),(2,1), and (3,0). The second case occurs when A + C = 2 and B + D = 13. In this case A = 0 and C = 2, in which (B,D) = (4,9), (5,8), (6,7), (7,6), (8,5), and (9,4), or A = 1 and C = 1, in which (B,D) = (4,9), (5,8), and (6,7), since B must be less than D. There are 16 possible ordered pairs for* $.\overline{AB}$ and $\overline{CD}$, *each equal to* $\frac{1}{3}$, *making the sum of all pairs* $\boxed{\frac{16}{3}}$.

2. *Using standard right triangle trigonometry,* $\cos\alpha=\frac{x}{AC}\Rightarrow AC=\frac{x}{\cos\alpha}$. *Since AC = CD and* $m\angle BAC=m\angle FCD=\alpha$, *the right triangles BAC and FCD are congruent and FC = x. Using right triangle trigonometry once more, we find*
$\tan FEC=\tan\alpha=\frac{x}{EF}\Rightarrow EF=\boxed{\frac{x}{\tan\alpha}}$.

3. *If f(x) has the roots* $3\pm4i$, *then c is the product of the roots, since the* x^2 *coefficient of f(x) is 1. Therefore,* $c=(3+4i)(3-4i)=25$, $f(x)=x^2-6x+25$. *g(x) is a linear equation and can be written as g(x) = mx + b. The equation* $f(x)=g(x)\Rightarrow x^2-6x+25=mx+b\Rightarrow x^2-(6+m)x+25-b=0$. *Since 0 is a solution to the equation f(x) = g(x), then* $(0)^2-(6+m)(0)+25-b=0\Rightarrow 25-b=0\Rightarrow b=25$. *The x value of the axis of symmetry of f(x) is* $-\frac{(-6)}{2(1)}=3$, *which is a solution to the equation f(x) = g(x). This means* $(3)^2-(6+m)(3)=0\Rightarrow 9-18-3m=0\Rightarrow m=-3$. *Therefore,* $\boxed{g(x)=-3x+25}$.

4. *See Figure 4. Draw* $\overline{SP}$ *and perpendicular segments to* $\overline{SP}$ *from vertices R and Q, creating right triangles SKR and PLQ. Using 45°-45°-90°* $\triangle$*SKR, we have SK = KR =* $\frac{5\sqrt{2}}{2}$. *Similarly,* $\triangle$*PLQ is also a 45°-45°-90° triangle, and PL = LQ =* $\frac{5\sqrt{2}}{2}$. *So SP = SK + KL + LP =* $5+5\sqrt{2}$. $\triangle$*SPV is also a 45°-45°-90° triangle, making SV =* $5+5\sqrt{2}$. *So, TV =5 + SV =* $\boxed{10+5\sqrt{2}}$.

5. *Let the acute angles be A and B. Since A + B = 90°,* $\sin B=\sin(90°-A)=\cos A$. *If we call the roots r = sin A and s = cos A, we know* $r^2+s^2=1$. *The equation* $17x^2-19x+k=0\Rightarrow x^2-\frac{19}{17}x+\frac{k}{17}=0$, *which means* $r+s=\frac{19}{17}$ *and* $rs=\frac{k}{17}$. *Since* $r^2+s^2=(r+s)^2-2rs$, *we have* $r^2+s^2=1=\left(\frac{19}{17}\right)^2-2\left(\frac{k}{17}\right)\Rightarrow 1=\frac{361}{289}-\frac{2k}{17}$. *Solving for k, we get k =* $\boxed{\frac{36}{17}=2\frac{2}{17}}$.

6. *Rewrite N as* $N=p^{2017}-4p^{2016}+4p^{2015}=p^{2015}(p^2-4p+4)=p^{2015}(p-2)^2$. *The value of p that will give us the fewest number of factors is 3. This means that* $N=3^{2015}(3-2)^2=3^{2015}$, *which has* $\boxed{2016}$ *factors.*