

Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

- _____ 1. If the shaded region shown in *Figure 1* can be described by the system $\begin{cases} x \leq 3 \\ \text{_____} \end{cases}$, write the missing inequality.



Figure 1

- _____ 2. John's age plus Debbie's age is 39. John is 3 years older than 3 times Debbie's age. How old is Debbie?

- $x =$ _____ 3. $A = \begin{bmatrix} x & y \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ 2 & 3 \end{bmatrix}$. If $\det A = 10$ and $\det B = 36$, determine x and y exactly.

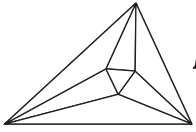
$y =$ _____

- $a =$ _____ 4. Determine exactly the necessary values of a and b in the linear equation $ax + by = -6$ so that the line passes through the intersection point of the lines $x + y = -4$ and $2x + 3y = -10$, and is also parallel to the line $-8x + 2y = 16$.

$b =$ _____

Name: _____

Team: _____



Minnesota State High School Mathematics League

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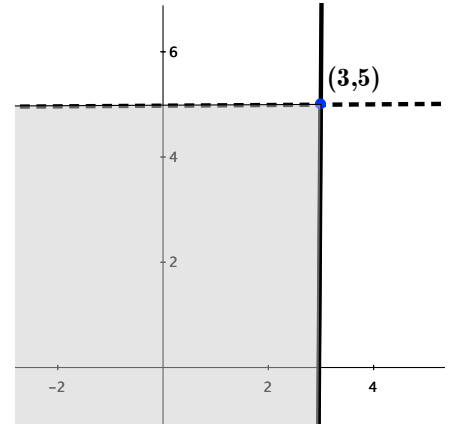
SOLUTIONS

$$y < 5$$

1. If the shaded region shown in *Figure 1* can be described by the system $\begin{cases} x \leq 3 \\ \underline{\hspace{2cm}} \end{cases}$, write the missing inequality.

$x \leq 3$ describes the solid vertical line. The dotted horizontal line is described by $y = 5$, and the shaded region is below it, so the missing inequality is $y < 5$.

Figure 1



$$9$$

2. John's age plus Debbie's age is 39. John is 3 years older than 3 times Debbie's age. How old is Debbie?

Let $D =$ Debbie's age. Then John's age can be described by $3 + 3D$. Since the sum of their ages is 39, $D + (3 + 3D) = 39 \Rightarrow 4D = 36 \Rightarrow D = 9$.

$$x = -160$$

$$y = -258$$

3. $A = \begin{bmatrix} x & y \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ 2 & 3 \end{bmatrix}$. If $\det A = 10$ and $\det B = 36$, determine x and y exactly.

$\det A = 8x - 5y = 10$. Likewise, $\det B = 3x - 2y = 36$. We now have a system of two linear equations: $\begin{cases} 8x - 5y = 10 \\ 3x - 2y = 36 \end{cases}$. Multiply the first equation by -2 and the second by 5 to get $\begin{cases} -16x + 10y = -20 \\ 15x - 10y = 180 \end{cases}$. Add the two equations to obtain $x = -160$; substitution yields $y = -258$.

$$a = 4$$

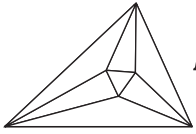
$$b = -1$$

4. Determine exactly the necessary values of a and b in the linear equation $ax + by = -6$ so that the line passes through the intersection point of the lines $x + y = -4$ and $2x + 3y = -10$, and is also parallel to the line $-8x + 2y = 16$.

First find the point of intersection between the lines $x + y = -4$ and $2x + 3y = -10$. Using either elimination or substitution, that point is $(-2, -2)$. Because the line $ax + by = -6$ passes through this point, substitute -2 for x and y to obtain the equation $-2a - 2b = -6$, i.e. $a + b = 3$. Also, $ax + by = -6$ is parallel to $-8x + 2y = 16$, so their slopes must be the same. This means $\frac{-a}{b} = 4 \Rightarrow a = -4b$. Substituting, $-4b + b = 3$, so $b = -1$ and $a = 4$.

Graders:
Award 1 point per correct value.

Graders:
Award 1 point per correct value.



Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

 cm 1. The surface area of a cube is 30 square centimeters.
Determine exactly the length of each of the cube's edges.

 2. A rectangle has integer side lengths, an area of 144, and a perimeter less than 60.
What is the maximum possible length of one of the rectangle's sides?

 TJ = 3. The side length of regular hexagon $FGHJKL$ is 8.
Let T be the midpoint of \overline{FL} (see *Figure 3*).
Determine exactly the length TJ .

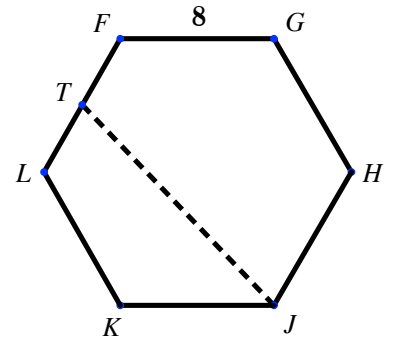


Figure 3

 VY = 4. Shown in *Figure 4* is a right trapezoidal prism where
 $WX = XY = YZ = 2$, and $WZ = 4$. If $VW = 5$, determine
exactly the length of diagonal \overline{VY} .

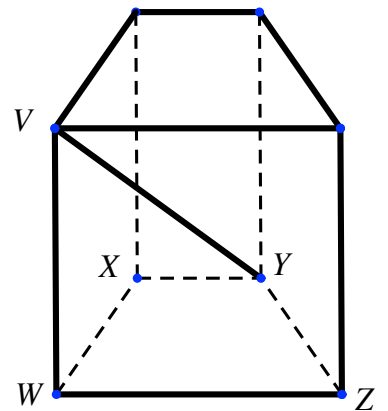
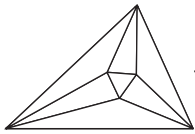


Figure 4

Name: _____

Team: _____



Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event B

SOLUTIONS

$$\boxed{\sqrt{5}} \text{ cm}$$

1. The surface area of a cube is 30 square centimeters. Determine exactly the length of each of the cube's edges.

Each of the cube's 6 faces has area $30 \div 6 = 5 \text{ cm}^2$, so each edge has a length of $\sqrt{5} \text{ cm}$.

$$\boxed{18}$$

2. A rectangle has integer side lengths, an area of 144, and a perimeter less than 60. What is the maximum possible length of one of the rectangle's sides?

Let the rectangle's length be L and its width be W , choosing $L \geq W$ for convenience. Perimeter less than 60 means that $2L + 2W < 60$, so $L + W < 30$. Area of 144 means that $LW = 144$, and because L and W are both integers, we are looking for pairwise factors of 144 whose sum is less than 30, with one factor as great as possible. Begin listing factors: $(L, W) = (144, 1), (72, 2), (48, 3), (36, 4), (24, 6) \dots$ before arriving at $L = 18, W = 8$.

$$TJ = \boxed{4\sqrt{13}}$$

3. The side length of regular hexagon $FGHJKL$ is 8. Let T be the midpoint of \overline{FL} (see Figure 3). Determine exactly the length TJ .

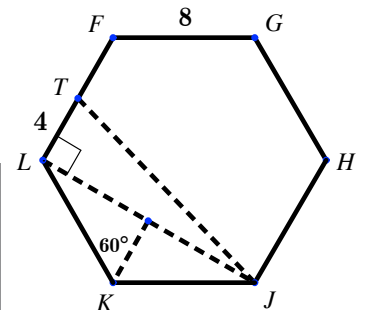
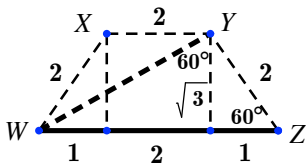


Figure 3

Draw LJ , creating right triangle TLJ and isosceles triangle LKJ . Drop an altitude from K , sectioning $\triangle LKJ$ into two congruent 30° - 60° - 90° triangles, each with hypotenuse 8 and leg lengths 4 and $4\sqrt{3}$. This gives $LJ = 8\sqrt{3}$, and by Pythagorean Theorem, $TJ = \sqrt{4^2 + (8\sqrt{3})^2} = \sqrt{16 + 192} = 4\sqrt{13}$.

$$VY = \boxed{\sqrt{37}}$$

4. Shown in Figure 4 is a right trapezoidal prism where $WX = XY = YZ = 2$, and $WZ = 4$. If $VW = 5$, determine exactly the length of diagonal \overline{VY} .



Draw WY , creating right triangle VWY . WY can be found by dropping altitudes from X and Y within trapezoid $WXYZ$ and using 30° - 60° - 90° triangles (see figure at left).

Then $VY = \sqrt{VW^2 + WY^2} = \sqrt{5^2 + (2\sqrt{3})^2} = \sqrt{25 + 12} = \sqrt{37}$.

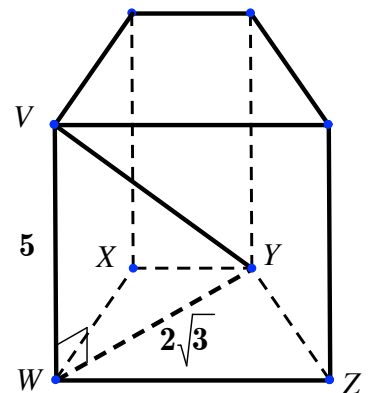
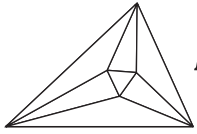


Figure 4



Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

_____ 1. Write the result of $(2 \cdot \cos 15^\circ + 2i \cdot \sin 15^\circ)^4$, in $(a + bi)$ form.

AC = _____ 2. In triangle ABC (Figure 2), $AB = 2$, $BC = 1$, and $m\angle ABC = 120^\circ$. Determine the length AC exactly.

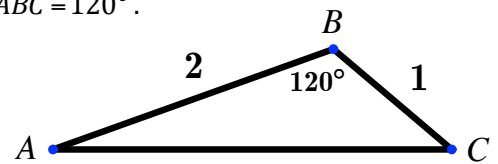


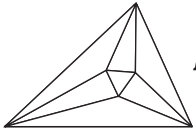
Figure 2

cos F = _____ 3. In triangle DEF , $DE = 3$, $EF = 4$, and $\angle D$ has a measure twice that of $\angle F$. Determine $\cos F$ exactly.

c = _____ 4. For $r > c$, the graphs of $f(x) = \cos^{-1} x$ and $x^2 + \left(y - \frac{\pi}{2}\right)^2 = r^2$ will not intersect. Determine exactly the least possible value of c .

Name: _____

Team: _____



Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$$\boxed{8+8\sqrt{3}\cdot i}$$

or $\boxed{8+8i\sqrt{3}}$

1. Write the result of $(2\cdot\cos 15^\circ+2i\cdot\sin 15^\circ)^4$, in $(a+bi)$ form.

Using DeMoivre's Theorem, $(2\cdot\cos 15^\circ+2i\cdot\sin 15^\circ)^4 = 2^4\cdot\cos(4\cdot 15^\circ)+2^4\cdot\sin(4\cdot 15^\circ)\cdot i$

$$= 16\cdot\cos 60^\circ+16\cdot\sin 60^\circ\cdot i = 16\cdot\frac{1}{2}+16\cdot\frac{\sqrt{3}}{2}\cdot i = 8+8\sqrt{3}\cdot i.$$

$$AC = \boxed{\sqrt{7}}$$

2. In triangle ABC (Figure 2), $AB = 2$, $BC = 1$, and $m\angle ABC = 120^\circ$. Determine the length AC exactly.

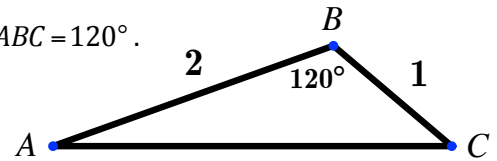


Figure 2

By the Law of Cosines, $AC^2 = 2^2 + 1^2 - 2\cdot 2\cdot 1\cdot\cos 120^\circ$

$$\Rightarrow AC = \sqrt{5 - 4\left(-\frac{1}{2}\right)} = \sqrt{7}.$$

$$\cos F = \boxed{\frac{2}{3}}$$

or $\boxed{0.\bar{6}}$

3. In triangle DEF , $DE = 3$, $EF = 4$, and $\angle D$ has a measure twice that of $\angle F$. Determine $\cos F$ exactly.

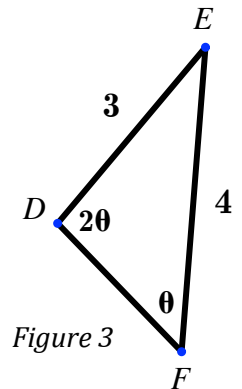


Figure 3

See Figure 3. Using the Law of Sines, $\frac{\sin \theta}{3} = \frac{\sin 2\theta}{4}$

$$\Rightarrow 4\sin \theta = 6\sin \theta \cos \theta \Rightarrow \sin \theta(4 - 6\cos \theta) = 0, \text{ so either } \sin \theta = 0$$

(impossible) or $4 - 6\cos \theta = 0 \Rightarrow \cos \theta = \frac{2}{3}$.

$$c = \boxed{\frac{\sqrt{4+\pi^2}}{2}}$$

4. For $r > c$, the graphs of $f(x) = \cos^{-1} x$ and $x^2 + \left(y - \frac{\pi}{2}\right)^2 = r^2$ will not intersect. Determine exactly the least possible value of c .

$x^2 + \left(y - \frac{\pi}{2}\right)^2 = r^2$ is a circle centered at $\left(0, \frac{\pi}{2}\right)$ with radius r .

When $r = c$, the circle will intersect $f(x) = \cos^{-1} x$ at the endpoints of f , $(-1, \pi)$ and $(1, 0)$. Thus c is the distance from $\left(0, \frac{\pi}{2}\right)$ to an endpoint: $c = \sqrt{1^2 + \left(\frac{\pi}{2}\right)^2} = \sqrt{1 + \frac{\pi^2}{4}} = \frac{\sqrt{4 + \pi^2}}{2}$.

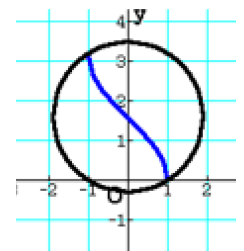
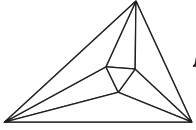


Figure 4



Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

_____ 1. Determine exactly the value of $9^{-\frac{1}{2}} + 8^{-\frac{1}{3}}$.

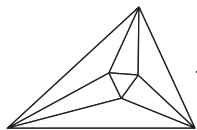
 x = _____ 2. Determine exactly all values of x for which $6(x+2)^2 = 14$.

_____ 3. For how many positive integers b is $\log_b 14 < \frac{1}{2} < \log_b 20$?

_____ 4. Find the product of all values n for which $\log_{16} n + \log_n 2 = \frac{5}{3}$.

Name: _____

Team: _____



Minnesota State High School Mathematics League

2014-15 Meet 3, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

$$\boxed{\frac{5}{6}}$$

1. Determine exactly the value of $9^{-\frac{1}{2}} + 8^{-\frac{1}{3}}$.

$$9^{-\frac{1}{2}} + 8^{-\frac{1}{3}} = \frac{1}{\sqrt{9}} + \frac{1}{\sqrt[3]{8}} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

$$x = \frac{-2 \pm \sqrt{21}}{3}$$

$$\text{or } \frac{-6 \pm \sqrt{21}}{3}$$

Graders:
Award 1 point
per correct value.

2. Determine exactly all values of x for which $6(x+2)^2 = 14$.

We could certainly expand the squared quantity on the left side, distribute 6, subtract 14, etc. to make use of the quadratic formula possible... but it is much more efficient to first divide both sides by six to get $(x+2)^2 = \frac{7}{3}$, then take the square root of both sides (don't forget the \pm sign!): $|x+2| = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3} \Rightarrow x+2 = \pm \frac{\sqrt{21}}{3} \Rightarrow x = -2 \pm \frac{\sqrt{21}}{3}$.

$$\boxed{203}$$

3. For how many positive integers b is $\log_b 14 < \frac{1}{2} < \log_b 20$?

$\log_b 14 < \frac{1}{2} < \log_b 20 \Rightarrow b^{\log_b 14} < b^{\frac{1}{2}} < b^{\log_b 20} \Rightarrow 14 < \sqrt{b} < 20$. Squaring all sides (and verifying that the inequality is still preserved in doing so), $196 < b < 400$. We are simply being asked to count the whole numbers between 196 and 400! There are 203 in all.

$$\boxed{64\sqrt[3]{4}} \text{ or } \boxed{2^{\frac{20}{3}}}$$

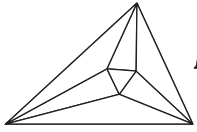
4. Find the product of all values n for which $\log_{16} n + \log_n 2 = \frac{5}{3}$.

Using change of base, $\log_{16} n + \log_n 2 = \frac{5}{3} \Rightarrow \frac{\log_2 n}{\log_2 16} + \frac{\log_2 2}{\log_2 n} = \frac{5}{3}$. Let $u = \log_2 n$. Then we

have $\frac{u}{4} + \frac{1}{u} = \frac{5}{3}$. Multiplying both sides by $12u$ results in the quadratic equation

$3u^2 + 12 = 20u \Rightarrow 3u^2 - 20u + 12 = 0 \Rightarrow (3u-2)(u-6) = 0$, so $u = \log_2 n = \frac{2}{3}$ or 6. Therefore,

$n = 2^{\frac{2}{3}}$ or $n = 2^6$, and the product of these values is $2^{\frac{2}{3}} \cdot 2^6 = 2^{\frac{20}{3}}$, or $64\sqrt[3]{4}$.



Minnesota State High School Mathematics League

2014-15 Meet 3, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

- _____ 1. Describe the set of all values of m for which the system $\begin{cases} mx + 3y = 9 \\ 3x + my = 9 \end{cases}$ will have exactly one solution (x, y) .

- Volume = _____ 2. *Figure 2* shows a cube of edge length 12. F and E are midpoints of \overline{DH} and \overline{CH} respectively. Calculate the volume of the solid whose vertices are $ABECDF$.

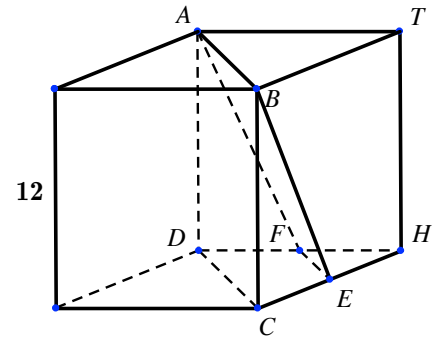


Figure 2

- _____ 3. If $\sin \alpha - \cos \alpha = \frac{1}{2}$, determine exactly the value of $\sin^4 \alpha + \cos^4 \alpha$.

- minutes _____ 4. Joey and Sonya, each eating at a constant rate, can devour an entire pizza together in 10 minutes. If Joey eats twice as quickly as Sonya, how long would it take Joey to eat half a pizza by himself?

- % _____ 5. An infinite floor is tiled with congruent regular octagons and squares as shown in *Figure 5*. Determine exactly the percentage of floor area covered by octagons.

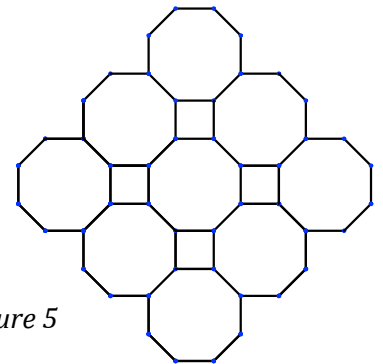
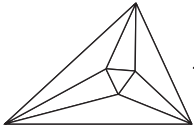


Figure 5

- Area = _____ 6. The side lengths of right triangle QRS measure $\log_2 b$, $\log_4 b$, and $\log_b 8$. Determine exactly the smallest possible area for $\triangle QRS$.

Team: _____



Minnesota State High School Mathematics League

2014-15 Meet 3, Team Event

SOLUTIONS (page 1)

$$m \neq \pm 3$$

or

$$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$

$$\{m < -3\} \cup \{-3 < m < 3\} \cup \{m > 3\}$$

or other equivalent description

1. Describe the set of all values of m for which the system $\begin{cases} mx + 3y = 9 \\ 3x + my = 9 \end{cases}$ will have exactly one solution (x, y) .

$$\text{Volume} = 360$$

2. Figure 2 shows a cube of edge length 12. F and E are midpoints of \overline{DH} and \overline{CH} respectively. Calculate the volume of the solid whose vertices are $ABECDF$.

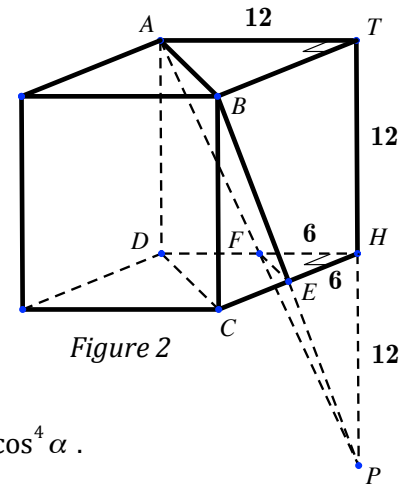


Figure 2

$$\frac{23}{32}$$

3. If $\sin \alpha - \cos \alpha = \frac{1}{2}$, determine exactly the value of $\sin^4 \alpha + \cos^4 \alpha$.

$$7.5 \text{ minutes}$$

$$\text{or } 7\frac{1}{2} \text{ or } \frac{15}{2}$$

4. Joey and Sonya, each eating at a constant rate, can devour an entire pizza together in 10 minutes. If Joey eats twice as quickly as Sonya, how long would it take Joey to eat half a pizza by himself?

$$200\sqrt{2} - 200 \%$$

5. An infinite floor is tiled with congruent regular octagons and squares as shown in Figure 5. Determine exactly the percentage of floor area covered by octagons.

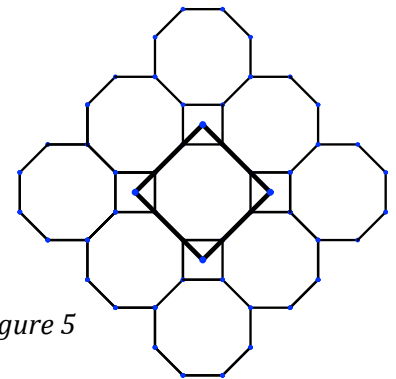
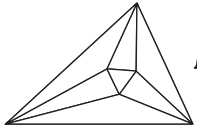


Figure 5

$$\frac{3\sqrt{5}}{10}$$

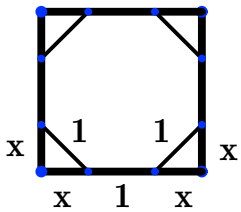
6. The side lengths of right triangle QRS measure $\log_2 b$, $\log_4 b$, and $\log_b 8$. Determine exactly the smallest possible area for $\triangle QRS$.



Minnesota State High School Mathematics League

2014-15 Meet 3, Team Event

SOLUTIONS (page 2)

1. Multiply the top equation by 3 and the bottom equation by m to yield $\begin{cases} 3mx + 9y = 27 \\ 3mx + m^2y = 9m \end{cases}$. Subtracting to eliminate x , $(9 - m^2)y = 27 - 9m \Rightarrow y = \frac{9(3-m)}{(3-m)(3+m)}$. y (and also x) will have a single value as long as $\boxed{m \neq \pm 3}$.
2. Extend \overline{TH} , \overline{BE} , and \overline{AF} until they meet at point P . The volume of pyramid $ABTP$ is $\frac{1}{3} \cdot \left(\frac{1}{2} \cdot 12^2\right) \cdot 24 = 576$. Meanwhile, the volume of pyramid $FEHP$ is $\frac{1}{3} \cdot \left(\frac{1}{2} \cdot 6^2\right) \cdot 12 = 72$, so the frustrum whose bases are ABT and FEH has volume $576 - 72 = 504$. Take the volume of half the cube (as sliced by plane $ABCD$), minus the volume of the frustrum: $864 - 504 = \boxed{360}$.
3. Square both sides of $\sin \alpha - \cos \alpha = \frac{1}{2}$ and rearrange terms to find that $\sin \alpha \cos \alpha = \frac{3}{8}$. Also, $(\sin^2 \alpha + \cos^2 \alpha)^2 = \sin^4 \alpha + 2\sin^2 \alpha \cos^2 \alpha + \cos^4 \alpha$, so $\sin^4 \alpha + \cos^4 \alpha = (\sin^2 \alpha + \cos^2 \alpha)^2 - 2\sin^2 \alpha \cos^2 \alpha = 1 - 2\left(\frac{3}{8}\right)^2 = 1 - \frac{9}{32} = \boxed{\frac{23}{32}}$.
4. Let Joey's eating rate (in pizzas/minute) be J . Then Sonya's is $\frac{1}{2}J$, since she eats at half of Joey's rate. The sum of their rates must then equal 1 pizza in 10 minutes; i.e., $J + \frac{1}{2}J = \frac{1}{10} \Rightarrow \frac{3}{2}J = \frac{1}{10} \Rightarrow J = \frac{1}{15}$. So Joey can eat a full pizza by himself in 15 minutes, meaning he would eat a half pizza in $\boxed{7.5}$ minutes.
5. Enclose one octagon in a square, as tightly as possible (see Figure 5). The area ratio of this octagon to its enclosing square is equal to that of all octagons to the entire floor. If the side length of the octagons and squares is 1, then the ratio is $\frac{(2x+1)^2 - 4\left(\frac{1}{2}x^2\right)}{(2x+1)^2}$. $x = \frac{1}{\sqrt{2}}$, so the ratio becomes $\frac{(\sqrt{2}+1)^2 - 1}{(\sqrt{2}+1)^2} = \frac{2+2\sqrt{2}}{3+2\sqrt{2}} \cdot \frac{3-2\sqrt{2}}{3-2\sqrt{2}} = 2\sqrt{2} - 2$. Multiply by 100: $\boxed{(200\sqrt{2} - 200)\%}$.
- 
6. Let $u = \log_2 b$. The side lengths are then u , $\frac{u}{2}$, and $\frac{3}{u}$. If $u > \sqrt{3}$, then u is the longest side; otherwise, $\frac{3}{u}$ is the longest side. Try both cases. In the first case, the area of $\triangle QRS$ is simply $\frac{1}{2} \cdot \frac{u}{2} \cdot \frac{3}{u} = \frac{3}{4}$. In the second case, the area is $\frac{1}{2} \cdot u \cdot \frac{u}{2} = \frac{u^2}{4}$, so we need the Pythagorean Theorem to show that $u^2 + \frac{u^2}{4} = \frac{9}{u^2} \Rightarrow \frac{5u^2}{4} = \frac{9}{u^2} \Rightarrow u^4 = \frac{36}{5} \Rightarrow u^2 = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$. Substituting, this results in an area of $\frac{6\sqrt{5}}{5} \div 4 = \boxed{\frac{3\sqrt{5}}{10}}$, which proves to be slightly smaller than $\frac{3}{4}$ (0.6708 versus 0.75).