# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

***NO CALCULATORS are allowed on this event.***

$x =$ ________ 1. Determine exactly the value of $x$ for which $2x-(1-3x)=-4$.

________ 2. What is the least integer value of $n$ that satisfies the inequality $\frac{n}{3}+4(n-1)>6$?

________ 3. Given that $4x + 7y = 6$, if the value of x increases by 3, determine exactly the amount by which the value of y decreases.

________ *gallons* 4. A hose providing a constant stream of water can fill a 30-gallon pool in 40 minutes. But because of a steady leak, it took the hose 50 minutes to fill the pool. How many gallons of water leaked out while the pool was being filled?

Name: ________________ Team: ________________

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event A

## SOLUTIONS

***NO CALCULATORS are allowed on this event.***

$x = \boxed{-\frac{3}{5}}$ **or** $\boxed{-0.6}$

1. Determine exactly the value of $x$ for which $2x-(1-3x)=-4$.

$$2x-(1-3x)=-4 \Rightarrow 2x-1+3x=-4 \Rightarrow 5x=-3 \Rightarrow x=-\frac{3}{5}.$$

$n = \boxed{3}$

2. What is the <u>least</u> <u>integer</u> value of $n$ that satisfies the inequality $\frac{n}{3}+4(n-1)>6$?

$\frac{n}{3}+4(n-1)>6 \Rightarrow \frac{13}{3}n>10 \Rightarrow n>\frac{30}{13}=2\frac{4}{13}$, *so the least integral solution is n = 3.*

$\boxed{\frac{12}{7}}$ **also accept** $\boxed{-\frac{12}{7}}$

3. Given that $4x + 7y = 6$, if the value of x increases by 3, determine exactly the amount by which the value of y <u>decreases</u>.

*Solve the given equation for y:* $y=\frac{6-4x}{7}$. *This is linear, so we can choose any two values for x and expect that the corresponding change in y will be the same regardless of our choice. Choose x = 0 and x = 3, for which y =* $\frac{6}{7}$ *and* $-\frac{6}{7}$ *respectively;* $\Delta y=\left(-\frac{6}{7}\right)-\left(\frac{6}{7}\right)=-\frac{12}{7}$.

$\boxed{7\frac{1}{2}}$ *gallons* **or** $\boxed{7.5}$

4. A hose providing a constant stream of water can fill a 30-gallon pool in 40 minutes. But because of a steady leak, it took the hose 50 minutes to fill the pool. How many gallons of water leaked out while the pool was being filled?

*If an empty 30-gallon pool with no leak takes 40 minutes to fill, then it is being filled at a rate of* $\frac{3}{4}$ *gallons/minute. If the same pool has a leak in it and takes 50 minutes to fill, then the fill rate has become* $\frac{3}{5}$ *gallons/minute. Therefore, water must have been leaking out at a rate of* $\frac{3}{4}-\frac{3}{5}=\frac{3}{20}$ *gallons/minute.* $50\ \text{min.}\cdot\frac{3\ \text{gallons}}{20\ \text{min.}}=7.5$ *total gallons leaked.*

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

*x* = ________________

1. In *Figure 1*, determine exactly the unknown length *x*.

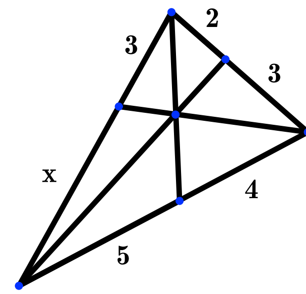



*Figure 1*

*CE* = ________________

2. *Figure 2* demonstrates what we might call the "Exterior Angle Bisector Theorem": In $\triangle ABC$, if $\overline{BD}$ bisects interior angle *ABC* and $\overline{BE}$ bisects exterior angle *CBF* (where *E* lies on the extension of side $\overline{AC}$ ), then $\frac{CE}{AE} = \frac{CD}{AD}$ . If $AD = 5$ and $CD = 2$, use this Theorem to determine *CE* exactly.

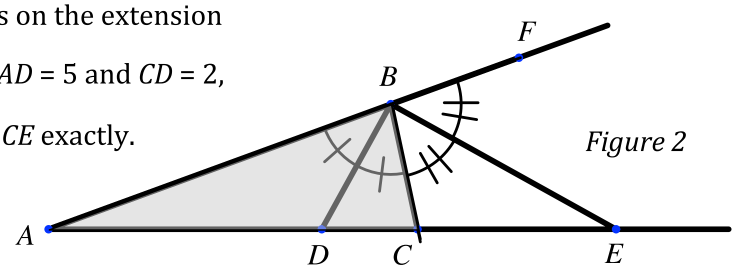



*Figure 2*

________________

3. The lengths of the sides of $\triangle PQR$ are 5, 12, and 13. Determine exactly the length of the triangle's shortest altitude.

*TZ* = ________________

4. Distinct points *W*, *X*, *Y*, and *Z* all lie on the same line such that $WX = XY = YZ$. For some point *T* <u>not</u> on that line, $TW = 3$, $TX = 4$, and $TY = 7$. Determine exactly the length *TZ*.

Name: ______________________________ Team: ______________________________

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event B

## SOLUTIONS

$x = \boxed{\dfrac{45}{8}}$ **or** $\boxed{5\dfrac{5}{8}}$ **or** $\boxed{5.625}$

1. In *Figure 1*, determine exactly the unknown length *x*.

*Using a straightforward treatment of Ceva's Theorem, we start at the segment labeled x and move clockwise around the triangle:* $\frac{x}{3}\cdot\frac{2}{3}\cdot\frac{4}{5}=1 \Rightarrow \frac{8x}{45}=1 \Rightarrow x=\frac{45}{8}$.

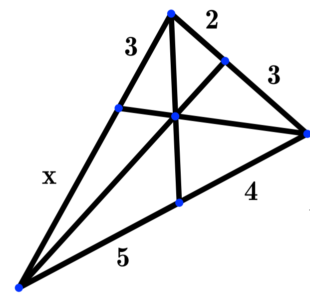


*Figure 1*

$CE = \boxed{\dfrac{14}{3}}$ **or** $\boxed{4\dfrac{2}{3}}$ **or** $\boxed{4.\overline{3}}$

2. *Figure 2* demonstrates what we might call the "Exterior Angle Bisector Theorem": In $\triangle ABC$, if $\overline{BD}$ bisects interior angle *ABC* and $\overline{BE}$ bisects exterior angle *CBF* (where *E* lies on the extension of side $\overline{AC}$), then $\frac{CE}{AE}=\frac{CD}{AD}$. If *AD* = 5 and *CD* = 2, use this Theorem to determine *CE* exactly.

$\frac{CE}{AE}=\frac{CD}{AD} \Rightarrow \frac{n}{n+7}=\frac{2}{5} \Rightarrow n=\frac{14}{3}$.

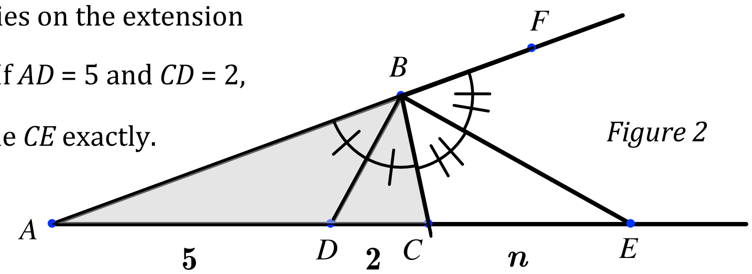


*Figure 2*

$\boxed{\dfrac{60}{13}}$ **or** $\boxed{4\dfrac{8}{13}}$

3. The lengths of the sides of $\triangle PQR$ are 5, 12, and 13. Determine exactly the length of the triangle's shortest altitude.

*The shortest altitude must be drawn to the triangle's longest side (13). The side lengths of 5, 12, and 13 make this a right triangle, so we can calculate its area in two ways (Figure 3): using 5 as the base and 12 as the height, for an area of 30; or using 13 as the base and* ***h*** *as the height, for an area of 13****h****/2. Set these quantities equal and solve to find that* ***h*** *= 60/13.*

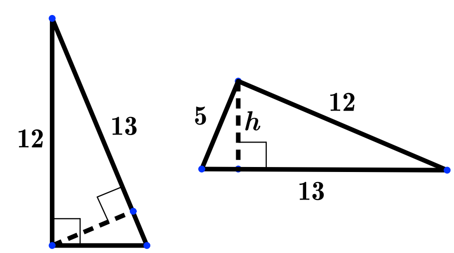


*Figure 3*

$TZ = \boxed{6\sqrt{3}}$

4. Distinct points *W*, *X*, *Y*, and *Z* all lie on the same line such that *WX* = *XY* = *YZ*. For some point *T* <u>not</u> on that line, *TW* = 3, *TX* = 4, and *TY* = 7. Determine exactly the length *TZ*.

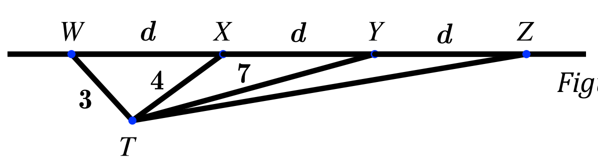


*Figure 4*

*See Figure 4. Apply Stewart's Theorem to* $\triangle TWY$: $3^2\cdot d+7^2\cdot d=4^2\cdot 2d+d\cdot d\cdot 2d \Rightarrow 58=32+2d^2$, *so* $d=\sqrt{13}$. *Now apply Stewart's again, this time to* $\triangle TXZ$: $4^2\cdot d+TZ^2\cdot d=7^2\cdot 2d+d\cdot d\cdot 2d$. *Divide through by d to yield* $16+TZ^2=98+2d^2$, *and we can use* $d=\sqrt{13}$ *to find that* $TZ^2=108$ *and* $TZ=6\sqrt{3}$.

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

***NO CALCULATORS are allowed on this event.***

$\sin 2\theta =$ __________

1. If $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{3}{5}$, determine exactly the value of $\sin 2\theta$.

$\cos(A - C) =$ __________

2. In right triangle $ABC$, $AB = 5$, $BC = 12$, and $AC = 13$. Determine $\cos(A - C)$ exactly.

$\alpha =$ __________

3. What is the <u>smallest</u> positive angle $\alpha$, in degrees, for which $\sqrt{2}\cdot\sin\alpha = \sqrt{1-\cos 218°}$ ?

$\beta =$ __________

4. What is the <u>smallest</u> positive angle $\beta$, in degrees, for which $\sin 2\beta - \tan\beta = \tan\beta\cdot\sin 58°$ ?

Name: ____________________ Team: ____________________

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event C

## SOLUTIONS

***NO CALCULATORS are allowed on this event.***

$\sin 2\theta = \boxed{-\frac{24}{25}}$ ***or*** $\boxed{-0.96}$

1. If $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{3}{5}$, determine exactly the value of sin 2*θ*.

*See Figure 1.* $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$.

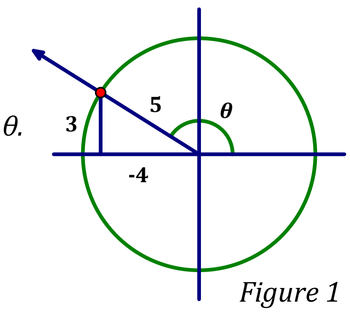
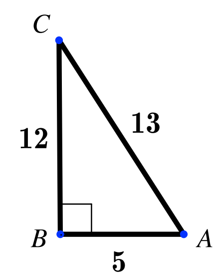



*Figure 1*

$\cos(A - C) = \boxed{\frac{120}{169}}$

2. In right triangle *ABC*, *AB* = 5, *BC* = 12, and *AC* = 13. Determine cos (*A* – *C*) exactly.

$\cos(A-C) = \cos A\cos C + \sin A\sin C = \left(\frac{5}{13}\right)\left(\frac{12}{13}\right) + \left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$.




$\alpha = \boxed{71°}$

3. What is the smallest positive angle $\alpha$, in degrees, for which $\sqrt{2}\cdot\sin\alpha = \sqrt{1-\cos 218°}$ ?

*Divide both sides of the equation by* $\sqrt{2}$ *to obtain* $\sin\alpha = \sqrt{\frac{1-\cos 218°}{2}}$. *The right side of this equation is the half-angle formula for sine, so we can rewrite:* $\sin\alpha = \sin 109°$. *Reflect* 109° *across the y-axis to find the smallest* $\alpha$ *with this sine value,* 71°.

$\beta = \boxed{16°}$

4. What is the smallest positive angle $\beta$, in degrees, for which $\sin 2\beta - \tan\beta = \tan\beta\cdot\sin 58°$ ?

*Rewrite the equation as* $2\sin\beta\cos\beta - \frac{\sin\beta}{\cos\beta} = \frac{\sin\beta}{\cos\beta}\cdot\sin 58°$. *Then multiply both sides by* $\frac{\cos\beta}{\sin\beta}$: $2\cos^2\beta - 1 = \sin 58° \Rightarrow \cos 2\beta = \sin 58°$. *But using the sine/cosine cofunction identity,* sin 58° = cos (90° – 58°) = cos 32°. *So* $\cos 2\beta = \cos 32°$, *and* $\beta = 16°$.

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

*slope* = ____________

1. Determine exactly the slope of the line $\sqrt{3}\cdot x - 2y = 6$ .

$(x, y)$ = ____________

2. Determine exactly the coordinates of the intersection point of $2x + 4y = 28$ and $\frac{x}{3}+\frac{y}{2}=1$ .

____________

3. A *lattice point* is a point on the *xy*-plane whose coordinates are both integers.
How many lattice points lie on the line $4x - 2y = 10$, are within the first quadrant, and have a $y$-coordinate of at most 2014?

____________

4. Determine exactly the shortest distance between the lines $2x - y = -6$ and $y = 2x + 15$.

Name: ____________________ Team: ____________________

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Individual Event D

## SOLUTIONS

slope = $\boxed{\dfrac{\sqrt{3}}{2}}$

1. Determine exactly the slope of the line $\sqrt{3}\cdot x-2y=6$.

*Place in slope-intercept form:* $\sqrt{3}\cdot x-2y=6 \Rightarrow 2y=\sqrt{3}\cdot x-6 \Rightarrow y=\dfrac{\sqrt{3}}{2}x-3$, *so the slope =* $\dfrac{\sqrt{3}}{2}$.

$(x, y)$ = $\boxed{(-30,22)}$

**Graders: Award 1 point per correct coordinate.**

2. Determine exactly the coordinates of the intersection point of $2x + 4y = 28$ and $\dfrac{x}{3}+\dfrac{y}{2}=1$.

*Multiply the second equation through by the LCD* (6) *so that both equations are in standard form:*

$$\begin{cases} 2x+4y=28 \\ 2x+3y=6 \end{cases}$$

*Using elimination,* $y=22 \Rightarrow 2x+4(22)=28 \Rightarrow x=-30$, *so the lines intersect at the point* (–30, 22).

$\boxed{1007}$

3. A *lattice point* is a point on the *xy*-plane whose coordinates are both integers. How many lattice points lie on the line $4x - 2y = 10$, are within the first quadrant, and have a $y$-coordinate of at most 2014?

*Figure 3 shows the graph of the line, which has x-intercept 2.5. As the line enters the first quadrant, it passes through the lattice points (3, 1), (4, 3), (5, 5), and so on. The y-coordinates are simply the set of odd numbers! So: we simply count the odd numbers ≤ 2014. This is the same as the # of even numbers ≤ 2014, which is 2014 ÷ 2 = 1007.*

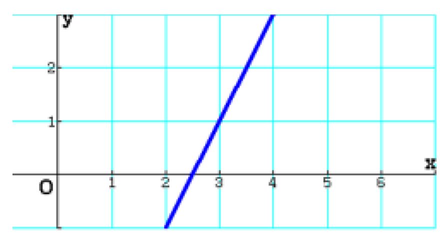



*Figure 3*

$\boxed{\dfrac{9\sqrt{5}}{5}}$

4. Determine exactly the shortest distance between the lines $2x - y = -6$ and $y = 2x + 15$.

*Rewriting the equation of the first line (y = 2x + 6), we can see that the lines have the same slope, meaning that they are parallel. Thus all distances between the two lines are equivalent (they are all "shortest"), and any distance between the two lines will suffice. y = 2x + 15 passes through (0, 15), so use the formula for the distance from a point to a line to find the distance from (0, 15) to 2x – y = –6 (Figure 4):*

$$d=\frac{|Ax+By-C|}{\sqrt{A^2+B^2}}=\frac{|2(0)+-1(15)-(-6)|}{\sqrt{2^2+(-1)^2}}=\frac{9}{\sqrt{5}}=\frac{9\sqrt{5}}{5}.$$

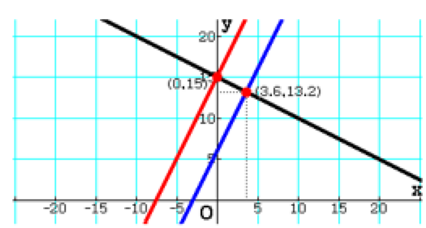



*Figure 4*

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

$\frac{F}{S} =$ ____________

1. On a father/son road trip, the father drove 60% of the time, and the son covered 60% of the distance while driving. Each drove at a constant rate (though not necessarily the same rate). If the father's rate was $F$ mph and the son's rate was $S$ mph, determine $\frac{F}{S}$ exactly.

____________

2. If the lengths of a triangle's sides are 4, 5, and 6, determine exactly the length of the triangle's <u>shortest</u> median.

$r =$ ____________

3. The line $2x - 3y = 6$ is tangent to a circle whose center lies on the positive $x$-axis. If the point of tangency is (6, 2), determine exactly the length of the circle's radius.

$x =$ ____________

$y =$ ____________

4. In *Figure 4*, the segment between $x$ and $y$ is an angle bisector. Determine exactly the unknown lengths $x$ and $y$.

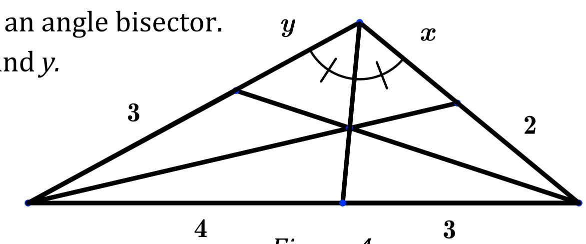



*Figure 4*

$k =$ ____________

5. Determine exactly the value of $k$ such that the $y$-intercept of $2ky - 5x = 9$ is 30 greater than the $y$-intercept of $2y - 5x = 9$.

$\sec\angle BFE =$ ____________

6. In right triangle $ABC$, the legs' measures are $AB = 6$ and $AC = 4$, and medians $\overline{BD}$ and $\overline{CE}$ intersect at $F$. Determine exactly $\sec\angle BFE$.

Team: ________________________

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Team Event

## SOLUTIONS (page 1)

$\frac{F}{S} = \boxed{\frac{4}{9}}$

1. On a father/son road trip, the father drove 60% of the time, and the son covered 60% of the distance while driving. Each drove at a constant rate (though not necessarily the same rate). If the father's rate was $F$ mph and the son's rate was $S$ mph, determine $\frac{F}{S}$ exactly.

$\boxed{\frac{\sqrt{46}}{2}}$

2. If the lengths of a triangle's sides are 4, 5, and 6, determine exactly the length of the triangle's <u>shortest</u> median.

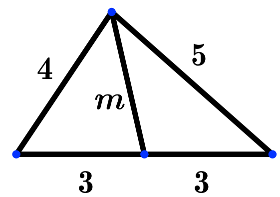



*Figure 2*

$\boxed{\frac{2\sqrt{13}}{3}}$

3. The line $2x - 3y = 6$ is tangent to a circle whose center lies on the positive $x$-axis. If the point of tangency is (6, 2), determine exactly the length of the circle's radius.

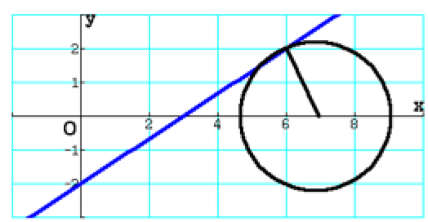



*Figure 3*

$x = \boxed{\frac{8}{5}}$ $y = \boxed{\frac{9}{5}}$

***Graders: award 2 points per correct value.***

4. In *Figure 4*, the segment between $x$ and $y$ is an angle bisector. Determine exactly the unknown lengths $x$ and $y$.

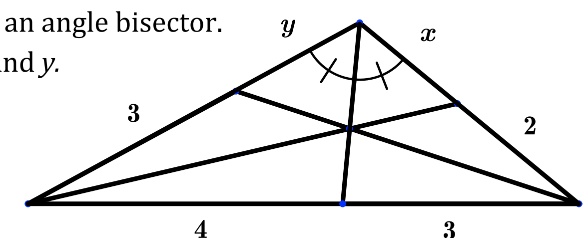



*Figure 4*

$k = \boxed{\frac{9}{2}}$

5. Determine exactly the value of $k$ such that the $y$-intercept of $2ky - 5x = 9$ is 30 greater than the $y$-intercept of $2y - 5x = 9$.

$\sec\angle BFE = \boxed{\frac{5\sqrt{10}}{13}}$

6. In right triangle $ABC$, the legs' measures are $AB = 6$ and $AC = 4$, and medians $\overline{BD}$ and $\overline{CE}$ intersect at $F$. Determine exactly $\sec\angle BFE$.

# Minnesota State High School Mathematics League

## 2014-15 Meet 2, Team Event

## SOLUTIONS (page 2)

1. *Suppose the trip covered a distance of* ***d*** *miles in a total time of* ***t*** *hours. Then the father drove a distance of 0.4****d*** *in 0.6****t*** *hours, for a rate of* $F=\frac{0.4d}{0.6t}=\frac{2d}{3t}$*, while the son drove a distance of 0.6****d*** *in 0.4****t*** *hours, for a rate of* $S=\frac{0.6d}{0.4t}=\frac{3d}{2t}$.
*Therefore,* $\frac{F}{S}=\frac{\frac{2d}{3t}}{\frac{3d}{2t}}=\frac{2d}{3t}\cdot\frac{2t}{3d}=\boxed{\frac{4}{9}}$. ***(Notice that the total distance and total time of the road trip were irrelevant!)***

2. *See Figure 2. Similar to what we saw in Event B, the shortest median must be drawn to the triangle's longest side (6).*
*By Stewart's Theorem,* $4^2\cdot3+5^2\cdot3=m^2\cdot6+3\cdot3\cdot6 \Rightarrow 48+75=6m^2+54 \Rightarrow m^2=\frac{69}{6}=\frac{23}{2} \Rightarrow m=\frac{\sqrt{23}}{\sqrt{2}}=\boxed{\frac{\sqrt{46}}{2}}$.

3. *See Figure 3. Rewrite the equation of the tangent line to find that it has slope* $\frac{2}{3}$*. Then the radius that passes through the point of tangency has slope* $-\frac{3}{2}$*, and is part of a line with equation* $y-2=-\frac{3}{2}(x-6)$*. This line's x-intercept is the circle's center:* $\left(\frac{22}{3},0\right)$*. r = the distance from this point to (6, 2):* $\sqrt{\left(6-\frac{22}{3}\right)^2+2^2}=\sqrt{\left(-\frac{4}{3}\right)^2+4}=\sqrt{\frac{16}{9}+\frac{36}{9}}=\boxed{\frac{2\sqrt{13}}{3}}$.

4. *First, apply the Angle Bisector Theorem:* $\frac{3+y}{2+x}=\frac{4}{3} \Rightarrow 9+3y=8+4x \Rightarrow 4x-3y=1$*. Then, by Ceva's Theorem,*
$\frac{3\cdot x\cdot 3}{y\cdot2\cdot4}=1 \Rightarrow \frac{9x}{8y}=1 \Rightarrow x=\frac{8}{9}y$*. Substituting,* $4\left(\frac{8}{9}y\right)-3y=1 \Rightarrow \frac{5}{9}y=1 \Rightarrow y=\boxed{\frac{9}{5}}$, $x=\frac{8}{9}y=\frac{8}{9}\left(\frac{9}{5}\right)=\boxed{\frac{8}{5}}$.

5. *Solve both equations for y:* $y=\frac{5}{2k}x+\frac{9}{2k}$ *and* $y=\frac{5}{2}x+\frac{9}{2}$*. Thus* $\frac{9}{2k}=30+\frac{9}{2} \Rightarrow 9=60k+9k \Rightarrow k=\boxed{\frac{3}{23}}$.

6. *Focus on* $\triangle BFE$ *in Figure 6.* $BD=\sqrt{2^2+6^2}=\sqrt{40}=2\sqrt{10}$*, so*
$\sin\angle EBF=\sin\angle ABD=\frac{AD}{BD}=\frac{2}{2\sqrt{10}}=\frac{\sqrt{10}}{10}$*, and* $\cos\angle EBF=\frac{3\sqrt{10}}{10}$.
*Also,* $CE=\sqrt{3^2+4^2}=5$ *and* $\angle BEF$ *and* $\angle AEC$ *are supplementary, so*
$\sin\angle BEF=\sin\angle AEC=\frac{AC}{CE}=\frac{4}{5}$ *and* $\cos\angle BEF=-\cos\angle AEC=-\frac{3}{5}$*. Finally:*
$m\angle BFE=180^\circ-(m\angle BEF+m\angle EBF) \Rightarrow \cos\angle BFE=-\cos(\angle BEF+\angle EBF)$
$\Rightarrow -(\cos\angle BEF\cdot\cos\angle EBF-\sin\angle BEF\cdot\sin\angle EBF)=-\left(-\frac{3}{5}\cdot\frac{3\sqrt{10}}{10}-\frac{4}{5}\cdot\frac{\sqrt{10}}{10}\right)$
$=\frac{9\sqrt{10}}{50}+\frac{4\sqrt{10}}{50}=\frac{13\sqrt{10}}{50}$*, and* $\sec\angle BFE=\frac{50}{13\sqrt{10}}=\frac{50\sqrt{10}}{130}=\boxed{\frac{5\sqrt{10}}{13}}$.

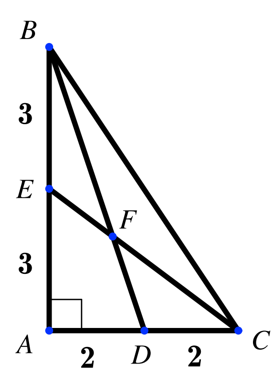



*Figure 6*