

# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

***NO CALCULATORS are allowed on this event.***

$$\frac{x}{y-z} =$$

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1. If  $x = \frac{1}{2}$ ,  $y = \frac{1}{3}$ , and  $z = \frac{1}{4}$ , determine exactly the value of  $\frac{x}{y-z}$ .

$$\frac{p}{q} =$$

---

2. Express 0.037037037... as a fraction  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime integers.

- \_\_\_\_\_ 3. Alice and Bob are the only income earners in their family. In 2012, Alice's salary grew by 20%, while Bob's salary dropped by 20%; however, their total family income was unchanged. In 2013, Alice's income again grew by 20%, while Bob's income again dropped by 20%. By what percentage did their family income change in 2013?  
*Express your answer as a percent, including a plus sign (+) to indicate an increase, or a minus sign (-) to indicate a decrease.*

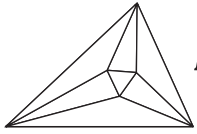
$$\max(a+c) =$$

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4. Let  $a$ ,  $b$ , and  $c$  be three positive integers. The greatest common divisor of  $a$  and  $b$  is 2; the greatest common divisor of  $b$  and  $c$  is 6; and the least common multiple of  $a$  and  $c$  is 72. Determine the maximum possible value of  $a + c$ .

Name: \_\_\_\_\_

Team: \_\_\_\_\_



# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event A

### SOLUTIONS

**NO CALCULATORS are allowed on this event.**

$$\frac{x}{y-z} = \boxed{6}$$

1. If  $x = \frac{1}{2}$ ,  $y = \frac{1}{3}$ , and  $z = \frac{1}{4}$ , determine exactly the value of  $\frac{x}{y-z}$ .

$$\frac{x}{y-z} = \frac{\frac{1}{2}}{\frac{1}{3} - \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{1}{12}} = \frac{1}{2} \cdot \frac{12}{1} = \frac{12}{2} = 6.$$

$$\frac{p}{q} = \boxed{\frac{1}{27}}$$

2. Express  $0.037037037\dots$  as a fraction  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime integers.

$$0.037037037\dots = 0.\overline{037} = \frac{37}{999} = \frac{37}{9 \cdot 111} = \frac{\cancel{37}}{9 \cdot 3 \cdot \cancel{37}} = \frac{1}{27}.$$

$$\boxed{+4\%}$$

**Graders:**

**Award 2 points only if + sign, "4", and % sign are given.**

**Award 1 point if one or both of + sign or % sign are missing.**

3. Alice and Bob are the only income earners in their family. In 2012, Alice's salary grew by 20%, while Bob's salary dropped by 20%; however, their total family income was unchanged. In 2013, Alice's income again grew by 20%, while Bob's income again dropped by 20%. By what percentage did their family income change in 2013? Express your answer as a percent, including a plus sign (+) to indicate an increase, or a minus sign (-) to indicate a decrease.

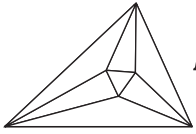
*We are not told Alice's or Bob's initial salaries, so that must not matter! If  $A$  and  $B$  represent Alice's and Bob's pre-2012 salaries, respectively, let's choose  $A = B = \$100$ . (Unrealistic for supporting a family, but nice values to work with percentage-wise.) Then, in 2012, Alice would earn  $\$120$ , and Bob would earn  $\$80$  (family income of  $\$120 + \$80 = \$200$  is unchanged). In 2013, Alice would earn  $\$120 \cdot 1.2 = \$144$ , while Bob would earn  $\$80 \cdot 0.8 = \$64$ . Their family income is now  $\$144 + \$64 = \$208$ , an increase of  $\frac{8}{200} = \frac{4}{100} = 4\%$ .*

$$\max(a+c) =$$

$$\boxed{80}$$

4. Let  $a$ ,  $b$ , and  $c$  be three positive integers. The greatest common divisor of  $a$  and  $b$  is 2; the greatest common divisor of  $b$  and  $c$  is 6; and the least common multiple of  $a$  and  $c$  is 72. Determine the maximum possible value of  $a + c$ .

*Because  $b$  is divisible by 3,  $a$  cannot be (otherwise their GCD would include a factor of 3). Since  $\text{LCM}(a, c) = 72$ ,  $a$  can contain a factor of 8, but not a factor of 16. Thus,  $a$  can only be 2, 4, or 8. If  $a = 2$  or 4, then  $c$  must be 72; if  $a = 8$ , then  $c$  could be 72, 36, or 18 (but not 9 ... why?). Our possible ordered pairs for  $(a, c)$  are  $(2, 72)$ ,  $(4, 72)$ ,  $(8, 72)$ ,  $(8, 36)$ , and  $(8, 18)$ , so  $a + c$  takes a maximum value of 80.*



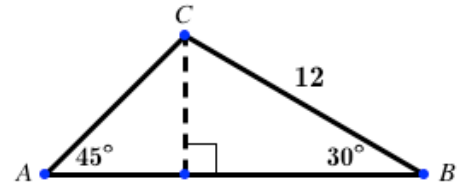
# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.  
Place your answer to each question on the line provided. You have 12 minutes for this event.

AC = \_\_\_\_\_

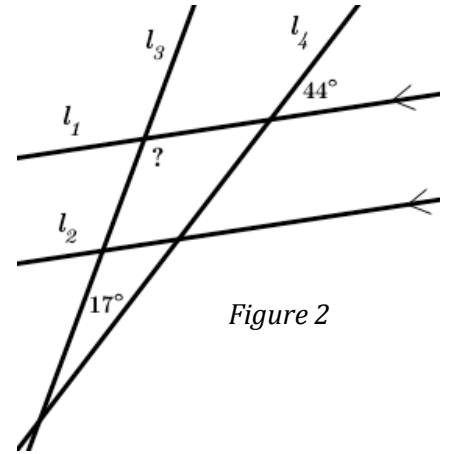
1. In  $\triangle ABC$ ,  $m\angle A = 45^\circ$  and  $m\angle B = 30^\circ$  as shown in *Figure 1*. If  $BC = 12$ , determine exactly the length  $AC$ .



*Figure 1*

\_\_\_\_\_

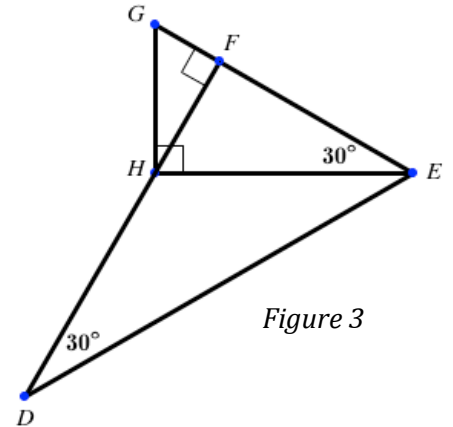
2. In *Figure 2*, lines  $l_1$  and  $l_2$  are parallel, while lines  $l_3$  and  $l_4$  intersect at an angle of  $17^\circ$ . If the acute angle formed by  $l_1$  and  $l_4$  measures  $44^\circ$ , calculate the measure of the obtuse angle between  $l_1$  and  $l_3$ .



*Figure 2*

DH = \_\_\_\_\_

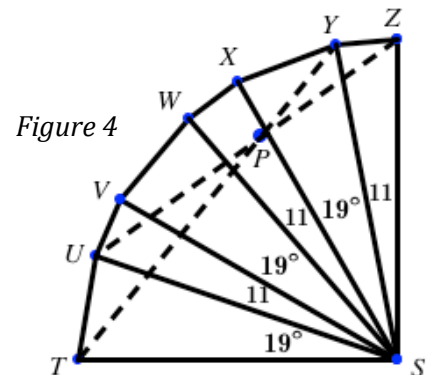
3. Triangles  $DEF$  and  $EGH$  are both right triangles, with  $m\angle D = m\angle GEH = 30^\circ$ , as shown in *Figure 3*. If  $EG = 12$ , determine exactly the length  $DH$ .



*Figure 3*

$m\angle TPZ =$  \_\_\_\_\_

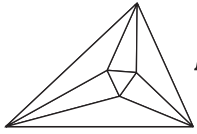
4. Three congruent isosceles triangles, each with apex angle  $11^\circ$ , are adjoined in alternating fashion with three other congruent isosceles triangles, each with apex angle  $19^\circ$ . This forms octagon  $STUVWXYZ$  (*Figure 4*). If  $\overline{TY}$  and  $\overline{UZ}$  intersect at  $P$ , calculate the measure of  $\angle TPZ$ .



*Figure 4*

Name: \_\_\_\_\_

Team: \_\_\_\_\_



# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event B

### SOLUTIONS

$$AC = \boxed{6\sqrt{2}}$$

1. In  $\triangle ABC$ ,  $m\angle A = 45^\circ$  and  $m\angle B = 30^\circ$  as shown in Figure 1. If  $BC = 12$ , determine exactly the length  $AC$ .

Label the foot of the altitude from  $C$  as  $D$ .  $\triangle BCD$  is  $30^\circ$ - $60^\circ$ - $90^\circ$  with hypotenuse 12, so  $CD = 6$ .  $\triangle ACD$  is an isosceles right triangle with leg 6, so  $AC = 6\sqrt{2}$ .

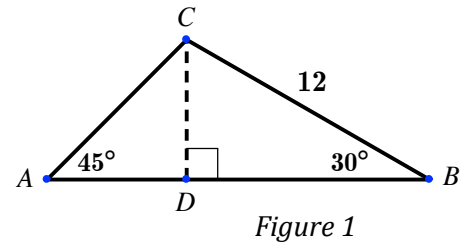


Figure 1

$$\boxed{119^\circ}$$

2. In Figure 2, lines  $\ell_1$  and  $\ell_2$  are parallel, while lines  $\ell_3$  and  $\ell_4$  intersect at an angle of  $17^\circ$ . If the acute angle formed by  $\ell_1$  and  $\ell_4$  measures  $44^\circ$ , calculate the measure of the obtuse angle between  $\ell_1$  and  $\ell_3$ .

Let your eyes find the largest triangle in the diagram. It has acute interior angles measuring  $17^\circ$  and  $44^\circ$ , so its obtuse angle measures  $180^\circ - 17^\circ - 44^\circ = 119^\circ$ .

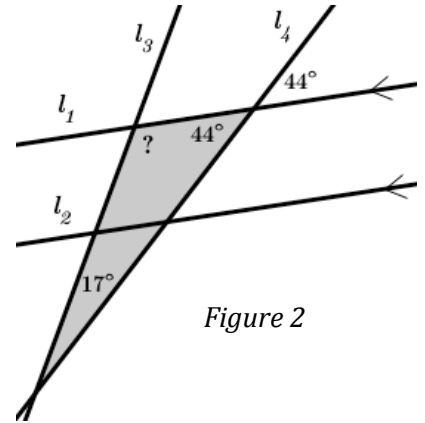


Figure 2

$$DH = \boxed{6\sqrt{3}}$$

3. Triangles  $DEF$  and  $EGH$  are both right triangles, with  $m\angle D = m\angle GEH = 30^\circ$ , as shown in Figure 3. If  $EG = 12$ , determine exactly the length  $DH$ .

Using  $\triangle EGH$ ,  $GH = 6$  and  $HE = 6\sqrt{3}$ .  $\triangle EFH$  is also  $30^\circ$ - $60^\circ$ - $90^\circ$ , with  $FH = 3\sqrt{3}$  and  $EF = 9$ . So, looking now at  $\triangle DEF$ ,  $DF = 9\sqrt{3}$ , and subtracting,  $DH = DF - FH = 6\sqrt{3}$ . (Notice that  $DH = EH$  ... so  $\triangle DHE$  is isosceles!)

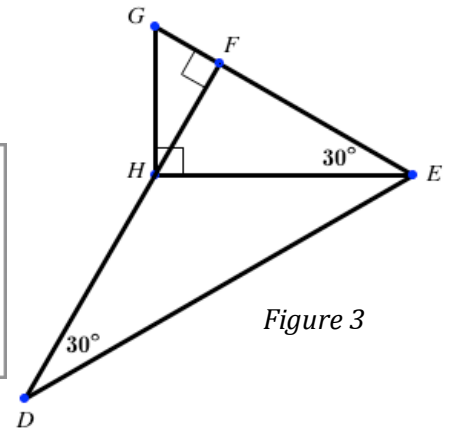


Figure 3

$$m\angle TPZ = \boxed{165^\circ}$$

4. Three congruent isosceles triangles, each with apex angle  $11^\circ$ , are adjoined in alternating fashion with three other congruent isosceles triangles, each with apex angle  $19^\circ$ . This forms octagon  $STUVWXYZ$  (Figure 4). If  $\overline{TY}$  and  $\overline{UZ}$  intersect at  $P$ , calculate the measure of  $\angle TPZ$ .

First off, it is critical to recognize that  $ST = SU = SV = SW = SX = SY = SZ$ . This means  $\triangle STY$  is isosceles, with apex angle  $79^\circ$  and vertex angles each  $(180^\circ - 79^\circ) \div 2 = 50.5^\circ$ , and also,  $\triangle SUZ$  is isosceles, with apex angle  $71^\circ$  and vertex angles each  $(180^\circ - 71^\circ) \div 2 = 54.5^\circ$ . Consider quadrilateral  $TPZS$ :  $m\angle T = 50.5^\circ$ ,  $m\angle S = 90^\circ$ , and  $m\angle Z = 54.5^\circ$ , so  $m\angle TPZ = 360^\circ - (50.5^\circ + 90^\circ + 54.5^\circ) = 165^\circ$ .

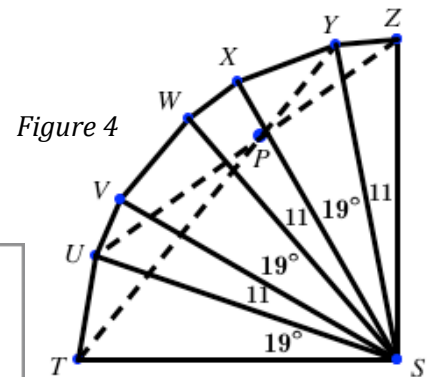
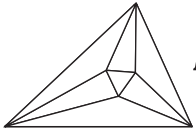


Figure 4



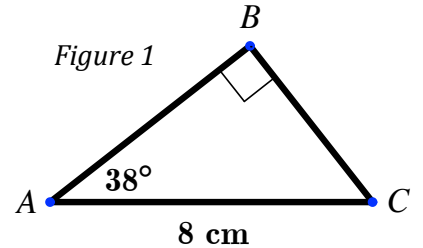
# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.  
Place your answer to each question on the line provided. You have 12 minutes for this event.

AB = \_\_\_\_\_ cm

1. *Figure 1* shows  $\triangle ABC$  with  $m\angle A = 38^\circ$  and  $AC = 8$  cm. Calculate the length of  $\overline{AB}$ .

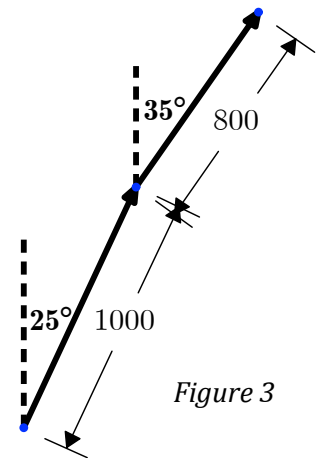


tan x = \_\_\_\_\_

2. If  $\sin x = -\frac{1}{3}$  and  $\pi < x < \frac{3\pi}{2}$ , determine exactly the value of  $\tan x$ .

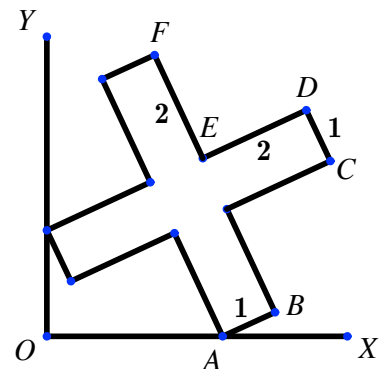
\_\_\_\_\_ yards

3. A man walks 1000 yards on a straight-line bearing of  $25^\circ$  east of north, and then 800 more yards on a straight-line bearing of  $35^\circ$  east of north. (see *Figure 3*.) Calculate how far north the man ends up from his starting point.



OA = \_\_\_\_\_

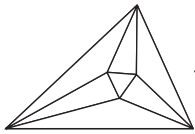
4. A symmetrical "plus" symbol, situated in Quadrant I, is tilted to rest against the  $x$ - and  $y$ -axes as shown in *Figure 4*. Given that  $DE = EF = 2$ ,  $AB = CD = 1$ , and  $m\angle BAX = 25^\circ$ , calculate the length  $OA$ .



*Figure 4*

Name: \_\_\_\_\_

Team: \_\_\_\_\_



# Minnesota State High School Mathematics League

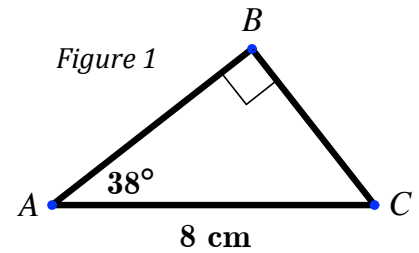
## 2014-15 Meet 1, Individual Event C

### SOLUTIONS

$$AB = \boxed{6.304} \text{ cm}$$

1. Figure 1 shows  $\triangle ABC$  with  $m\angle A = 38^\circ$  and  $AC = 8$  cm. Calculate the length of  $\overline{AB}$ .

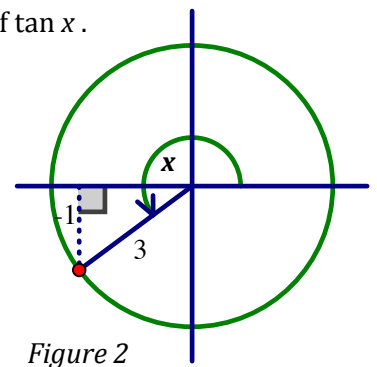
$$\cos 38^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{AB}{8} \Rightarrow AB = 8 \cdot \cos 38^\circ \approx 6.304.$$



$$\tan x = \boxed{\frac{\sqrt{2}}{4}}$$

2. If  $\sin x = -\frac{1}{3}$  and  $\pi < x < \frac{3\pi}{2}$ , determine exactly the value of  $\tan x$ .

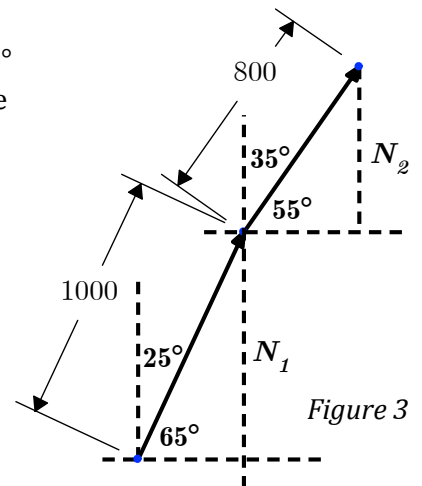
See Figure 2, where a reference triangle has been created using a circle of radius 3. By the Pythagorean Theorem, the horizontal leg has length  $\sqrt{3^2 - (-1)^2} = \sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2}$ , and because it extends in the negative  $x$  direction, we assign it the value  $-2\sqrt{2}$ . Thus  $\tan x = \frac{\text{opp}}{\text{adj}} = \frac{-1}{-2\sqrt{2}} = \frac{\sqrt{2}}{4}$ .



$$\boxed{1561.629} \text{ yards}$$

3. A man walks 1000 yards on a straight-line bearing of  $25^\circ$  east of north, and then 800 more yards on a straight-line bearing of  $35^\circ$  east of north. (see Figure 3.) Calculate how far north the man ends up from his starting point.

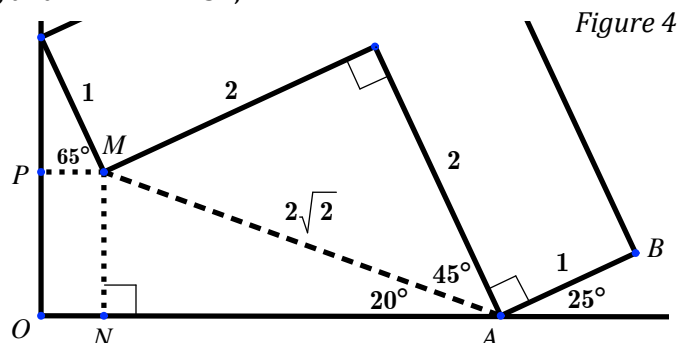
Construct right triangles as shown in Figure 3. Then we have  $\sin 65^\circ = \frac{N_1}{1000}$  and  $\sin 55^\circ = \frac{N_2}{800}$ , and the man's northward displacement is  $N_1 + N_2 = 1000 \sin 65^\circ + 800 \sin 55^\circ \approx 1561.629$ .

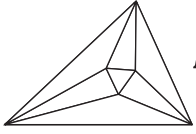


$$OA = \boxed{3.080}$$

4. A symmetrical "plus" symbol, situated in Quadrant I, is tilted to rest against the  $x$ - and  $y$ -axes as shown in Figure 4. Given that  $DE = EF = 2$ ,  $AB = CD = 1$ , and  $m\angle BAX = 25^\circ$ , calculate the length  $OA$ .

Label point  $M$  as shown in the figure. Then draw  $\overline{MA}$ , creating an isosceles right triangle with  $MA = 2\sqrt{2}$ . Draw perpendiculars from  $M$  meeting the axes at  $N$  and  $P$ . Then  $NA = 2\sqrt{2} \cdot \cos 20^\circ$ , and  $PM = ON = 1 \cdot \cos 65^\circ$ , so  $OA = ON + NA = \cos 65^\circ + 2\sqrt{2} \cdot \cos 20^\circ \approx 3.080$ .





# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

***NO CALCULATORS are allowed on this event.***

\_\_\_\_\_ 1. Determine exactly the sum of the roots of the cubic polynomial  $2x^3 - 9x^2 + 14x - 6$ .

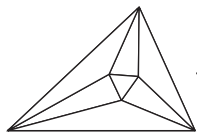
$k =$  \_\_\_\_\_ 2. Determine exactly the value of  $k$  for which the two solutions of  $3x^2 - 4x + k = 0$  are equal.

$N =$  \_\_\_\_\_ 3. Let  $N$  be a positive two-digit number. If the sum of the squares of its digits is equal to 100 times its tens digit minus twice its ones digit, what is  $N$ ?

$x =$  \_\_\_\_\_ 4. Determine exactly all real solutions of  $x^2 - x = \frac{1}{x^2 - x}$ .

Name: \_\_\_\_\_

Team: \_\_\_\_\_



# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Individual Event D

### SOLUTIONS

***NO CALCULATORS are allowed on this event.***

$$\boxed{\frac{9}{2}} \text{ or } \boxed{4\frac{1}{2}}$$

$$\text{or } \boxed{4.5}$$

1. Determine exactly the sum of the roots of the cubic polynomial  $2x^3 - 9x^2 + 14x - 6$ .

The sum of the roots is the opposite of the  $x^2$  coefficient, divided by the leading coefficient:  $\frac{9}{2}$ .

$$k = \boxed{\frac{4}{3}} \text{ or } \boxed{1\frac{1}{3}}$$

$$\text{or } \boxed{1.\bar{3}}$$

2. Determine exactly the value of  $k$  for which the two solutions of  $3x^2 - 4x + k = 0$  are equal.

When a quadratic equation has two equal roots, the value of its discriminant is 0. Therefore,  
 $(-4)^2 - 4 \cdot 3 \cdot k = 0 \Rightarrow k = \frac{4}{3}$ .

$$N = \boxed{19}$$

3. Let  $N$  be a positive two-digit number. If the sum of the squares of its digits is equal to 100 times its tens digit minus twice its ones digit, what is  $N$ ?

Let  $\underline{A}$  be the tens digit of  $N$ , and let  $\underline{B}$  be the ones digit.  $N$  can then be represented by  $N = 10A + B$ , with  $A \geq 1$ . According to the relationship described in the problem,  $A^2 + B^2 = 100A - 2B$ . Looking at the left side of this relationship, the largest possible value of  $A^2 + B^2$  is  $9^2 + 9^2 = 162$ . But looking at the right side of the relationship, if  $100A - 2B \leq 162$ , then  $A$  could not be greater than 1; also, an  $A$ -value of 2 or greater would give  $100A - 2B$  a minimum value of 182. Therefore,  $A = 1$ , and  $1 + B^2 = 100 - 2B \Rightarrow B^2 + 2B - 99 = 0 \Rightarrow (B + 11)(B - 9) = 0$ , so  $B = 9$ , and  $N = 19$ .

$$x = \boxed{\frac{1 \pm \sqrt{5}}{2}}$$

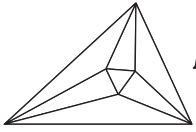
$$\text{or } \boxed{\frac{1 \pm \sqrt{5}}{2}}$$

4. Determine exactly all real solutions of  $x^2 - x = \frac{1}{x^2 - x}$ .

Let  $y = x^2 - x$ . Then  $y = \frac{1}{y} \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ . So either  $x^2 - x = -1 \Rightarrow x^2 - x + 1 = 0$ , which has a discriminant of  $(-1)^2 - 4(1)(1) = -3$  and therefore no real roots, or  $x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$ , which has discriminant  $(-1)^2 - 4(1)(-1) = 5$ , and real solutions  $x = \frac{1 \pm \sqrt{5}}{2}$ .

**Graders:**  
Award only 1 point if any extra solutions are given, or for only one correct solution.





# Minnesota State High School Mathematics League

## 2014-15 Meet 1, Team Event

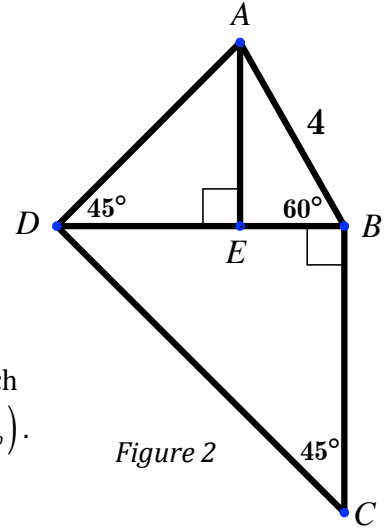
Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

(p, q) = \_\_\_\_\_

1. Two positive integers  $p$  and  $q$  have a greatest common divisor of 6 and a least common multiple of 120. If  $p < q$ , list all possible ordered pairs  $(p, q)$ .

$AC^2 =$  \_\_\_\_\_

2. In *Figure 2*, isosceles right triangles  $ADE$  and  $BCD$  are attached to  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle  $ABE$  (where  $AE > BE$ ), creating quadrilateral  $ABCD$ . If  $AB = 4$ , determine exactly the value of  $AC^2$ .



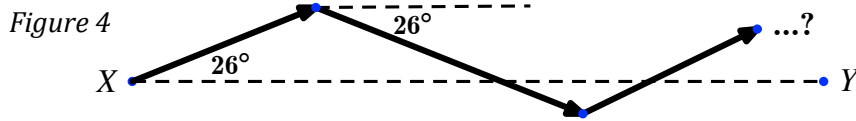
*Figure 2*

(a, b) = \_\_\_\_\_

3. List one possible ordered pair of positive integers  $(a, b)$  such that 243 in base  $a$  ( $\underline{243}_a$ ) is equal to 1123 in base  $b$  ( $\underline{1123}_b$ ).

\_\_\_\_\_ yds

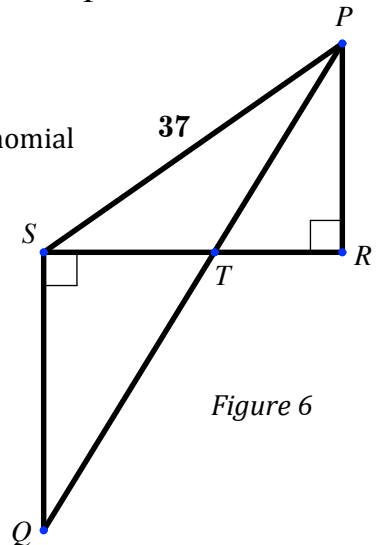
4. Fighting a strong wind, a boat sails from  $X$  to a location  $Y$  (due east of  $X$ ), alternating course between bearings of  $26^\circ$  north of east and  $26^\circ$  south of east (see *Figure 4*). If the boat travels a total of 5000 yards, calculate the straight-line distance from  $X$  to  $Y$ .



*Figure 4*

$k =$  \_\_\_\_\_

5. What is the integer value of  $k$  for which the cubic polynomial  $x^3 - 5x^2 + kx + 9$  has two equal rational roots?

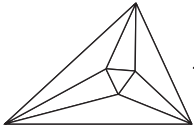


*Figure 6*

$PQ =$  \_\_\_\_\_

6. In *Figure 6*, segments  $\overline{PQ}$  and  $\overline{RS}$  intersect at  $T$ . All seven line segments in the figure have integer side lengths. If  $PS = 37$ , determine exactly the largest possible length for  $\overline{PQ}$ .

Team: \_\_\_\_\_



# Minnesota State High School Mathematics League

2014-15 Meet 1, Team Event

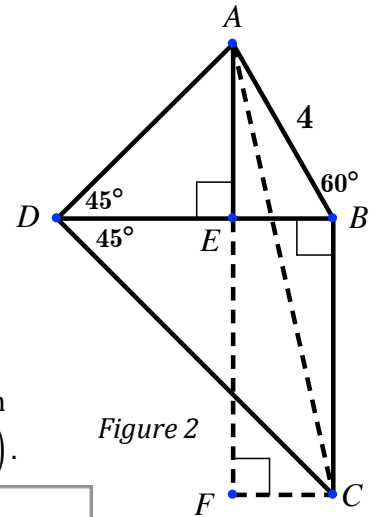
## SOLUTIONS (page 1)

$$(p, q) = \boxed{(6, 120) \text{ or } (24, 30)}$$

*Graders: award 2 points per correct value given; deduct 1 point for each incorrect value.*

$$AC^2 = \boxed{56 + 16\sqrt{3}}$$

2. In *Figure 2*, isosceles right triangles  $ADE$  and  $BCD$  are attached to  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle  $ABE$  (where  $AE > BE$ ), creating quadrilateral  $ABCD$ . If  $AB = 4$ , determine exactly the value of  $AC^2$ .



$$(a, b) = \boxed{(8, 5)}$$

$$\text{or } \boxed{(84, 24)}$$

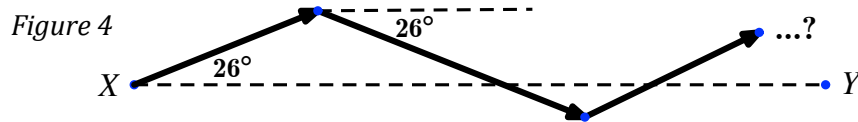
*or others*

3. List one possible ordered pair of positive integers  $(a, b)$  such that 243 in base  $a$  ( $\underline{243}_a$ ) is equal to 1123 in base  $b$  ( $\underline{1123}_b$ ).

*Graders: use the fact that  $2a(a+2) = b(b^2 + b + 2)$  to verify student answers.*

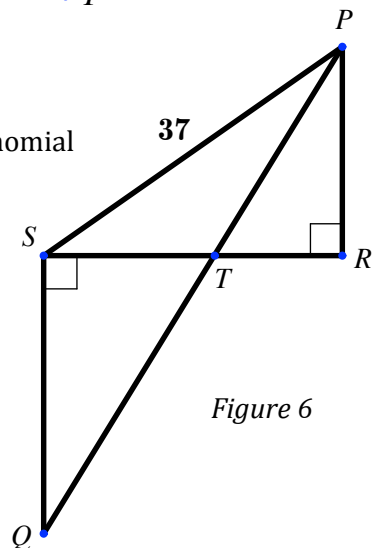
$$\boxed{4493.970} \text{ yds}$$

4. Fighting a strong wind, a boat sails from  $X$  to a location  $Y$  (due east of  $X$ ), alternating course between bearings of  $26^\circ$  north of east and  $26^\circ$  south of east (see *Figure 4*). If the boat travels a total of 5000 yards, calculate the straight-line distance from  $X$  to  $Y$ .



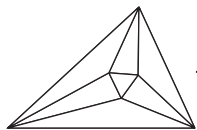
$$k = \boxed{3}$$

5. What is the integer value of  $k$  for which the cubic polynomial  $x^3 - 5x^2 + kx + 9$  has two equal rational roots?



$$PQ = \boxed{91}$$

6. In *Figure 6*, segments  $\overline{PQ}$  and  $\overline{RS}$  intersect at  $T$ . All seven line segments in the figure have integer side lengths. If  $PS = 37$ , determine exactly the largest possible length for  $\overline{PQ}$ .



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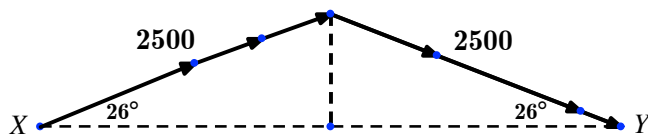
### SOLUTIONS (page 2)

1. Let  $p = 6m$  and  $q = 6n$ . Because  $pq = \text{GCD}(p,q) \cdot \text{LCM}(p,q)$ , we have  $(6m)(6n) = 6 \cdot 120 \Rightarrow mn = 20$ .  $m < n$ , so the possible ordered pairs  $(m, n)$  are  $(1, 20)$ ,  $(2, 10)$ , and  $(4, 5) \dots$  but be careful! If  $m = 2, n = 10$ , then  $p$  and  $q$  will share another factor of 2, and their GCD would be 12, not 6.  $(m, n) = (1, 20)$  or  $(4, 5) \Rightarrow (p, q) = \boxed{(6, 120) \text{ or } (24, 30)}$ .

2. See Figure 2. After drawing  $\overline{AC}$ , extend  $\overline{AE}$  to meet the perpendicular from  $C$  at  $F$ , creating right triangle  $AFC$ . Using  $30^\circ\text{-}60^\circ\text{-}90^\circ$  triangle  $ABE$ , we have  $EB = FC = 2$ ,  $AE = DE = 2\sqrt{3}$ , and  $EF = BC = DE + EB = 2\sqrt{3} + 2$ . So  $AF = AE + EF = 4\sqrt{3} + 2$ , and by the Pythagorean Theorem,  $AC^2 = AF^2 + FC^2 = (4\sqrt{3} + 2)^2 + 2^2 = (48 + 16\sqrt{3} + 4) + 4 = \boxed{56 + 16\sqrt{3}}$ .  
**( $\triangle ADC$  is also a right triangle, and can be used with  $AD = 2\sqrt{6}$  and  $DC = 2\sqrt{6} + 2\sqrt{2}$  to obtain the same result.)**

3. If  $(\underline{243}_a) = (\underline{1123}_b)$ , then  $2a^2 + 4a + 3 = b^3 + b^2 + 2b + 3 \Rightarrow 2a^2 + 4a = b^3 + b^2 + 2b \Rightarrow 2a(a+2) = b(b^2 + b + 2)$ . We have some additional clues: (1) the greatest digit appearing in the base- $a$  representation of 243 is 4, so  $a \geq 5$ ; (2) the greatest digit appearing in the base- $b$  representation of 1123 is 3, so  $b \geq 4$ ; and (3)  $(\underline{1123}_b)$  occupies more place values than  $(\underline{243}_a)$ , so  $b < a$ . One possible solution technique is to use increasing values of  $a$ , starting with 5, and see if the resulting cubic equation in  $b$  has an integer solution with  $b \geq 4$ :  $2(5)^2 + 4(5) = b^3 + b^2 + 2b \Rightarrow b^3 + b^2 + 2b - 70 = 0 \Rightarrow b \approx 3.665$ ;  $a = 6 \Rightarrow b^3 + b^2 + 2b - 96 = 0 \Rightarrow b \approx 4.134$ ;  $a = 7 \Rightarrow b^3 + b^2 + 2b - 126 = 0 \Rightarrow b \approx 4.577$ ;  $a = 8 \Rightarrow b^3 + b^2 + 2b - 160 = 0 \Rightarrow b = 5$ ; so one possible pair  $(a, b)$  is  $\boxed{(8, 5)}$ .  **$(a, b) = (84, 24)$  also works, as may others.**

4. No matter how many turns the boat takes, it will eventually have to cover an equal distance bearing north of east as it does bearing south of east. This is because we can treat the boat's individual movements as vectors, rearranging them to form the isosceles triangle shown in the figure below.  $\cos 26^\circ = \frac{\frac{1}{2}(XY)}{2500} \Rightarrow XY = 5000 \cos 26^\circ \approx \boxed{4493.970}$  yards.



5. By the Rational Roots Theorem, the only possible rational roots of  $x^3 - 5x^2 + kx + 9$  are  $\pm 1, 3$ , and 9. For each possibility, use synthetic division, set the remainder equal to 0 to determine the value of  $k$  necessary for the  $x$ -value to be a root, and then see if that value of  $k$  results in a quadratic polynomial that either has the same root or is a perfect square itself:

$$\begin{array}{r|rrrr} 1 & 1 & -5 & k & 9 \\ & & 1 & -4 & k-4 & \boxed{k+5} \end{array} \Rightarrow k = -5 \Rightarrow \cancel{x^2 - 4x - 9} = 0 \quad \begin{array}{r|rrrr} -1 & 1 & -5 & k & 9 \\ & & 1 & -6 & k+6 & \boxed{3-k} \end{array} \Rightarrow k = 3 \Rightarrow x^2 - 6x + 9 = (x-3)^2 = 0,$$

so  $k = \boxed{3}$  causes the original cubic to have two equal rational roots ( $x = 3$ , multiplicity 2).

6. The only Pythagorean triple with hypotenuse 37 is 12/35/37. The figure may not be to scale; i.e., we don't know whether  $SR > PR$  or vice versa. If  $PR = 35$ , then  $\triangle PRT$  causes us to seek Pythagorean triples with one leg of length 35 and the other leg  $RT < 12$  and as short as possible (so that  $\overline{PQ}$  will be as long as possible). Unfortunately, such a triple would have to be of the form  $x/35/36$ , and does not exist. So  $PR = 12$  and  $SR = 35$ . Now in  $\triangle PRT$ , we need triples with one leg of length 12, and the other leg  $RT < 35$  and as short as possible. Hopefully 5/12/13 comes to mind, so that  $RT = 5, ST = 30$ , and  $PT = 13$ . Finally, in  $\triangle STQ$ , we seek the triple with short leg 30 and longest possible hypotenuse:  $30/72/78 \Rightarrow PT + TQ = \boxed{91}$ .