Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$X =$ ________________ 1. The product of the repeating decimals 0.33333 . . . and 0.66666 . . . is the repeating decimal $0.\underline{X}\,\underline{X}\,\underline{X}\,\underline{X}\,\underline{X}\ldots$ What digit does X represent?

________________ 2. Write the base-ten number 140 in base 15.

________________ 3. A used-car dealer sold two cars, receiving $560 in payment for each car. One of those transactions resulted in a 40% profit for the dealer, while the other resulted in a 20% loss. What was the dealer's net profit (in dollars) for the two transactions?

________________ 4. Write a six-digit number whose first three digits are $\underline{6}\,\underline{3}\,\underline{7}\ldots$ and which is divisible by 21, 22, 23, and 24.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event A

SOLUTIONS

$X =$ 2

1. The product of the repeating decimals $0.33333\ldots$ and $0.66666\ldots$ is the repeating decimal $0.\underline{X}\underline{X}\underline{X}\underline{X}\underline{X}\ldots$ What digit does X represent? ***[The 1st HS Math League Problem Book]***

$(0.33333\ldots)(0.66666\ldots)=\frac{1}{3}\cdot\frac{2}{3}=\frac{2}{9}=0.22222\ldots$ *, so X = 2.*

95

(Student does <u>not</u> necessarily need to include the subscript "15" to indicate base.)

2. Write the base-ten number 140 in base 15.

$140 \div 15 = 9R5$*, so there are 9 fifteens and 5 ones in 140. Thus,* $140_{10} = 95_{15}$.

$20

(Also accept "20")

3. A used-car dealer sold two cars, receiving $560 in payment for each car. One of those transactions resulted in a 40% profit for the dealer, while the other resulted in a 20% loss. What was the dealer's net profit (in dollars) for the two transactions? ***[Mathematics Teacher]***

The first car was sold for 140% of its cost to the dealer, so we divide $560 by 1.4 to find that the car's cost was $400. Dealer's profit on first car = $560 - $400 = $160.
The second car was sold for 80% of its cost, so we divide $560 by 0.8: car's cost was $700.
Dealer's profit on second car = $560 - $700 = -$140, or a $140 loss.
Net profit on both cars = $160 - $140 = $20.

637560

(Do <u>not</u> accept "560". The problem clearly states that a six-digit number is to be written.)

4. Write a six-digit number whose first three digits are $\underline{6}\underline{3}\underline{7}\ldots$ and which is divisible by 21, 22, 23, and 24. ***[Mathematics Teacher]***

$21=3\cdot7,\ 22=2\cdot11,\ 23=\textit{prime}$*, and* $24=2^3\cdot3\ldots$ *so* $LCM(21,22,23,24)=2^3\cdot3\cdot7\cdot11\cdot23=42504$.
Since this number is so large, only one multiple of it can possibly be in the "637 thousands".
Divide 637000 by 42504 to get approximately 14.99, so the necessary multiplier is 15:

$$42504\cdot15=637560 .$$

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

______________ 1. *Figure 1* shows a star-like object formed by overlaying two isosceles triangles: $\triangle ABC$ with apex angle $A = 40°$, and $\triangle DEF$ with apex angle $D = 50°$.
Calculate $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E + m\angle F$.

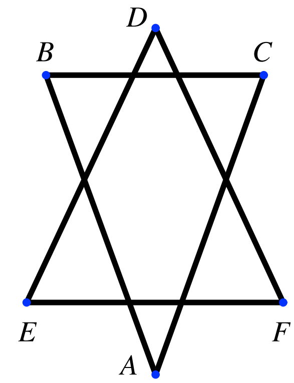

Figure 1

______________ 2. Beginning with regular pentagon $A_1A_2A_3A_4A_5$, we form five-pointed star $S_1S_2S_3S_4S_5$ by extending the edges of the pentagon, as shown in *Figure 2*.
Calculate $m\angle S_1 + m\angle S_2 + m\angle S_3 + m\angle S_4 + m\angle S_5$, where S_1, S_2, S_3, S_4, S_5 are the <u>acute</u> angles at each vertex.

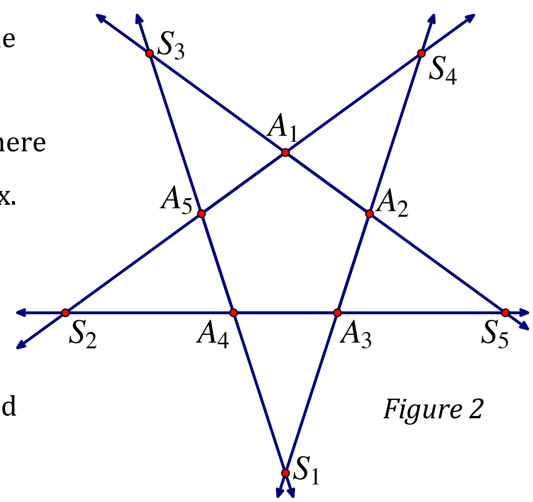

Figure 2

PT = ______________ 3. In right triangle *PQR (Figure 3)*, *PQ* = 2, *PR* = 4, and $\overline{PS}$ bisects $\angle P$. A line drawn through *R*, perpendicular to $\overline{QR}$, meets the extension of $\overline{PS}$ at T.
Determine exactly the length of $\overline{PT}$.

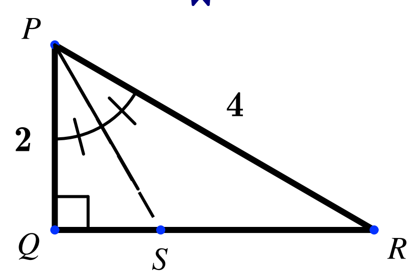

Figure 3

r = ______________ 4. An isosceles triangle with legs of length 50 and a base of length 60 is inscribed in a circle.
Calculate the length of the circle's radius.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event B

SOLUTIONS

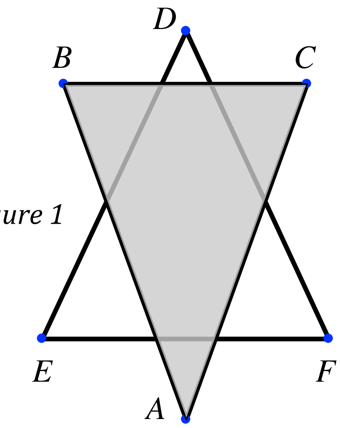

Figure 1

360°

1. *Figure 1* shows a star-like object formed by overlaying two isosceles triangles: $\triangle ABC$ with apex angle $A = 40°$, and $\triangle DEF$ with apex angle $D = 50°$.
Calculate $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E + m\angle F$.

You are really just being asked for the sum of the interior angles of each of two triangles: $\triangle ABC$ (shaded) and $\triangle DEF$. Since each triangle's angles sum to 180°, the overall sum is 360°. (Sometimes you don't need all of the given information.)

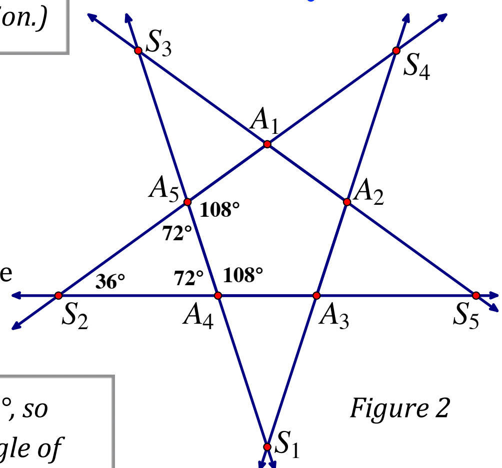

Figure 2

180°

2. Beginning with regular pentagon $A_1A_2A_3A_4A_5$, we form five-pointed star $S_1S_2S_3S_4S_5$ by extending the edges of the pentagon, as shown in *Figure 2*.
Calculate $m\angle S_1 + m\angle S_2 + m\angle S_3 + m\angle S_4 + m\angle S_5$, where S_1, S_2, S_3, S_4, S_5 are the <u>acute</u> angles at each vertex.

Each interior angle of the regular pentagon measures 108°, so their supplements measure 72°, as shown. Each vertex angle of the star is also the vertex angle of a 36°-72°-72° isosceles triangle. The sum of those vertex angles is 5(36°) = 180°.

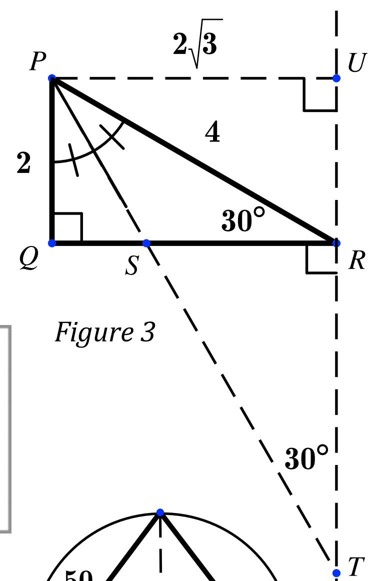

Figure 3

PT = **$4\sqrt{3}$**

3. In right triangle *PQR (Figure 3)*, *PQ* = 2, *PR* = 4, and $\overline{PS}$ bisects $\angle P$. A line drawn through *R*, perpendicular to $\overline{QR}$, meets the extension of $\overline{PS}$ at *T*. Determine exactly the length of $\overline{PT}$.

$\triangle PQR$ is 30°-60°-90°, so $QR = 2\sqrt{3}$. Draw $\overline{UP}$ perpendicular to $\overleftrightarrow{RT}$ at point U. $\overline{UP}$ also has length $2\sqrt{3}$, and is part of 30°-60°-90° triangle PTU. $\overline{PT}$ is the hypotenuse of $\triangle PTU$, with length $4\sqrt{3}$.

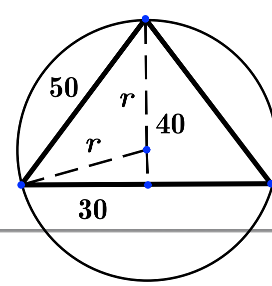

Figure 4

r = **31.25**

4. An isosceles triangle with legs of length 50 and a base of length 60 is inscribed in a circle.
Calculate the length of the circle's radius.

The triangle can be split in half, forming two 30-40-50 triangles. (Figure 4)
Drawing convenient radii reveals that $30^2 + (40 - r)^2 = r^2 \Leftrightarrow 900 + 1600 - 80r + \cancel{r^2} = \cancel{r^2} \Leftrightarrow r = 31.25$.

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

$\sin\beta =$ ____________ 1. In the right triangle shown in *Figure 1*, $\tan\alpha = 1$. Determine exactly the value of $\sin\beta$.

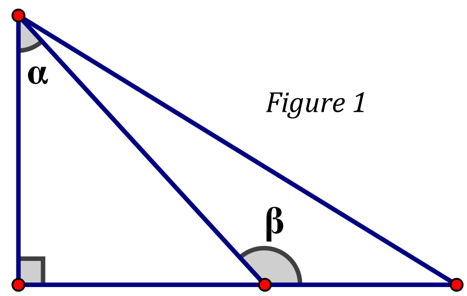

Figure 1

$\tan\phi =$ ____________ 2. If $\sin\phi = \dfrac{5}{13}$, determine exactly all possible values for $\tan\phi$.

$\sin C =$ ____________ 3. In $\triangle ABC$ *(Figure 3)*, $\overline{BD}$ is drawn perpendicular to hypotenuse $\overline{AC}$ so that the lengths AD and CD are in the ratio 1 : 4. Determine exactly the value of sin C.

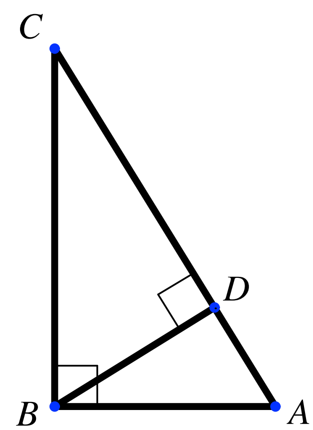

Figure 3

____________ 4. Calculate the sum of all values of $\cos\left(\dfrac{m}{n}\pi\right)$, where m and n can each equal 1, 2, 3, or 4.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\sin\beta = \boxed{\frac{\sqrt{2}}{2}}$

1. In the right triangle shown in *Figure 1*, $\tan\alpha = 1$. Determine exactly the value of $\sin\beta$.

$\tan\alpha = 1 \Rightarrow \alpha = 45°$, *so* $\sin\beta = \sin 135° = \frac{\sqrt{2}}{2}$.

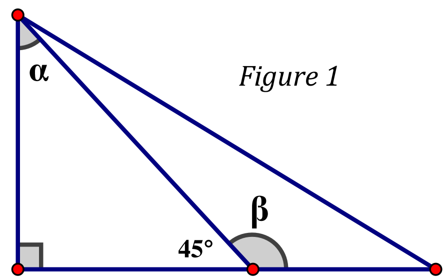

Figure 1

$\tan\phi = \boxed{\pm\frac{5}{12}}$

(Graders: award one point if only one sign is listed.)

2. If $\sin\phi = \frac{5}{13}$, determine exactly all possible values for $\tan\phi$.

Creating a reference triangle for ϕ *(Figure 2), we see that* $\sin\phi$ *can equal* $\frac{5}{13}$ *in Quadrants I and II. The reference triangle is a 5-12-13 triangle, so* $\tan\phi = \pm\frac{5}{12}$.

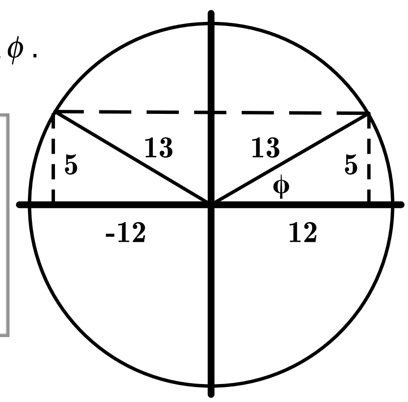

Figure 2

$\sin C = \boxed{\frac{\sqrt{5}}{5}}$

3. In $\triangle ABC$ *(Figure 3)*, $\overline{BD}$ is drawn perpendicular to hypotenuse $\overline{AC}$ so that the lengths *AD* and *CD* are in the ratio 1 : 4. Determine exactly the value of sin *C*.

Label AB = x, AD = n, and CD = 4n, as shown. Note that nested similar right triangles cause $\angle C$ *and* $\angle DBA$ *to be congruent, so that* $\sin\angle C = \frac{x}{5n} = \sin\angle DBA = \frac{n}{x} \Rightarrow x^2 = 5n^2 \Rightarrow x = n\sqrt{5}$.

Substituting, $\sin\angle C = \frac{\cancel{n}\sqrt{5}}{5\cancel{n}} = \frac{\sqrt{5}}{5}$.

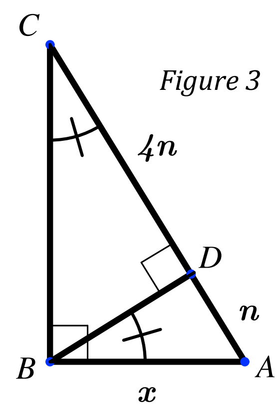

Figure 3

$\boxed{-\frac{5}{2}}$ **or** $\boxed{-2\frac{1}{2}}$ **or** $\boxed{-2.5}$

4. Calculate the sum of all values of $\cos\left(\frac{m}{n}\pi\right)$, where *m* and *n* can each equal 1, 2, 3, or 4.

$\left(\cos\frac{\pi}{1} + \cos\frac{2\pi}{1}\right)$, $\left(\cos\frac{3\pi}{1} + \cos\frac{4\pi}{1}\right)$, $\left(\cos\frac{2\pi}{2} + \cos\frac{4\pi}{2}\right)$, $\left(\cos\frac{\pi}{3} + \cos\frac{2\pi}{3}\right)$, *and* $\left(\cos\frac{\pi}{4} + \cos\frac{3\pi}{4}\right)$ *are all pairs that sum to zero. Of the remaining values,* $\cos\frac{\pi}{2}$, $\cos\frac{3\pi}{2}$, *and* $\cos\frac{2\pi}{4}$ *are all themselves equal to zero. The desired sum is* $\cos\frac{3\pi}{3} + \cos\frac{4\pi}{3} + \cos\frac{4\pi}{4} = (-1) + \left(-\frac{1}{2}\right) + (-1) = -\frac{5}{2}$.

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

_______________ 1. Express $(x+1)(x+10)+(x+4)(x-4)$ as the product of two binomials, each with integer coefficients.

$c =$ _______________ 2. What is the **greatest** integer c for which the quadratic polynomial $5x^2+11x+c$ has two distinct rational roots?

$k =$ _______________ 3. What is the only integer k for which the cubic polynomial $x^3-7x^2+kx+15$ can be factored into three linear binomials, each with integer coefficients?

$f(5) =$ _______________ 4. Alec found that a quadratic polynomial $f(x)$ had zeroes at -1 and -11. He forgot the values of the polynomial's coefficients, but remembered that its discriminant was 64 and that $f(10)$ was less than $f(-10)$. Calculate $f(5)$.

Name: ____________________________ Team: ____________________________

Minnesota State High School Mathematics League

2012-13 Meet 1, Individual Event D

SOLUTIONS

$\boxed{(2x-1)(x+6)}$

(Graders: Obviously, listing these factors in the opposite order is still correct.)

1. Express $(x+1)(x+10)+(x+4)(x-4)$ as the product of two binomials, each with integer coefficients.

$$(x+1)(x+10)+(x+4)(x-4)=x^2+11x+10+x^2-16=2x^2+11x-6=(2x-1)(x+6).$$

$c = \boxed{6}$

2. What is the **greatest** integer c for which the quadratic polynomial $5x^2+11x+c$ has two distinct rational roots?

For there to be two distinct rational roots, the discriminant must be a perfect square greater than zero. Evaluate b^2-4ac *:* $11^2-4(5)(c)=121-20c$. *The smallest possible nonzero perfect square is 1, which occurs when c = 6.*

$k = \boxed{7}$

3. What is the only integer k for which the cubic polynomial $x^3-7x^2+kx+15$ can be factored into three linear binomials, each with integer coefficients?

Since the leading coefficient is 1, we can express the factorization as $(x-p)(x-q)(x-r)$, *where p, q, and r are the three roots of the polynomial. Sum of roots = p + q + r = 7, while product of roots = pqr = −15. Since p, q, and r must be integers, the product must be formed by the factor triples* (1, 1, 15) *or* (1, 3, 5), *with either one or three negative signs within each triple. The only triple whose members sum to 7 is* $(p,q,r)=(-1,3,5) \Rightarrow k=(-1)(3)+(3)(5)+(-1)(5)=7$.

$f(5) = \boxed{-76.8}$ *or* $\boxed{-\frac{384}{5}}$ *or* $\boxed{-76\frac{4}{5}}$

4. Alec found that a quadratic polynomial $f(x)$ had zeroes at −1 and −11. He forgot the values of the polynomial's coefficients, but remembered that its discriminant was 64 and that $f(10)$ was less than $f(-10)$. Calculate $f(5)$.

$f(x)=a(x+1)(x+11)=ax^2+12ax+11a$. *Since the discriminant of this quadratic is 64, we have:* $(12a)^2-4(a)(11a)=144a^2-44a^2=100a^2=64 \Rightarrow a^2=\frac{64}{100} \Rightarrow a=\pm\frac{8}{10}=\pm\frac{4}{5}$. *Using the factored form,* $f(10)=\pm\frac{4}{5}(11)(21)$ *and* $f(-10)=\pm\frac{4}{5}(-9)(1)$, *so the only way for* $f(10)$ *to be less than* $f(-10)$ *is if a is negative; i.e.,* $a=-\frac{4}{5}$. *Thus,* $f(5)=-\frac{4}{5}(5+1)(5+11)=-\frac{4}{5}(6)(16)=-\frac{384}{5}$.

Minnesota State High School Mathematics League

2012-13 Meet 1, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

______________ 1. Find the sum of all values of $\cos\left(\frac{m}{n}\pi\right)$, where m and n are integers such that $1 \le m \le n \le 2012$.

$b =$ ______________ 2. In some number base b, the number 121 is equal to the decimal (base-10) number 324. Calculate b.

$a =$ ______________ 3. Determine exactly all values of a for which $y = \left(a^2+1\right)\left(x^2-1\right) - ax + 7$ has two real zeroes whose sum equals their product.

$(x, y, z) =$ ______________ 4. The fraction $\frac{37}{13}$ can be written in the form $2 + \cfrac{1}{x + \cfrac{1}{y + \cfrac{1}{z}}}$,

where x, y, and z are positive integers. Find the values of (x, y, z).

$c =$ ______________ 5. For what value of c can the polynomial $x^{17} - 7x^{16} + \ldots + cx + 17$ be factored completely into 17 linear binomials, each with integer coefficients?
(Note: in this problem, the ellipsis [. . .] is not intended to suggest any kind of pattern, only the fact that the interior terms of the polynomial are unknown.)

______________ 6. Beginning with an arbitrary n-sided polygon $A_1A_2\ldots A_n$, we form an n-pointed star $S_1S_2\ldots S_n$ by extending edges $\overline{A_nA_1}$ and $\overline{A_3A_2}$ to form point S_1, extending $\overline{A_1A_2}$ and $\overline{A_4A_3}$ to form point S_2, and so on. In terms of n, determine exactly $m\angle S_1 + m\angle S_2 + \ldots + m\angle S_n$.

Team: ______________________________

Minnesota State High School Mathematics League

2012-13 Meet 1, Team Event

SOLUTIONS (page 1)

$\boxed{-2012}$ 1. Find the sum of all values of $\cos\left(\frac{m}{n}\pi\right)$, where m and n are integers such that $1 \le m \le n \le 2012$.

$b = \boxed{17}$ 2. In some number base b, the number 121 is equal to the decimal (base-10) number 324. Calculate b. ***[Mathematics Teacher]***

$a = \boxed{-3}$ 3. Determine exactly all values of a for which $y = (a^2+1)(x^2-1) - ax + 7$ has two real zeroes whose sum equals their product.

Graders: Award only 2 pts if extraneous a = 2 is also listed

$(x, y, z) = \boxed{(1,5,2)}$ 4. The fraction $\frac{37}{13}$ can be written in the form $2 + \cfrac{1}{x + \cfrac{1}{y + \cfrac{1}{z}}}$,

where x, y, and z are positive integers. Find the values of (x, y, z). ***[Australian M.C., 1981]***

$c = \boxed{169}$ 5. For what value of c can the polynomial $x^{17} - 7x^{16} + \ldots + cx + 17$ be factored completely into 17 linear binomials, each with integer coefficients?

(Note: in this problem, the ellipsis [. . .] is not intended to suggest any kind of pattern, only the fact that the interior terms of the polynomial are unknown.)

$\boxed{180°(n-4)}$ or $\boxed{180n - 720}$ 6. Beginning with an arbitrary n-sided polygon $A_1A_2\ldots A_n$, we form an n-pointed star $S_1S_2\ldots S_n$ by extending edges $\overline{A_nA_1}$ and $\overline{A_3A_2}$ to form point S_1, extending $\overline{A_1A_2}$ and $\overline{A_4A_3}$ to form point S_2, and so on. In terms of n, determine exactly $m\angle S_1 + m\angle S_2 + \ldots + m\angle S_n$.

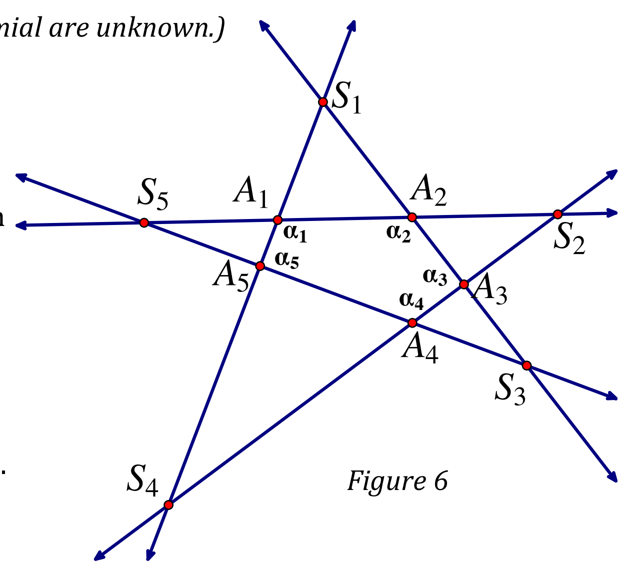

Figure 6

Minnesota State High School Mathematics League

2012-13 Meet 1, Team Event

SOLUTIONS (page 2)

1. *From facts like* $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ *and* $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$*, we quickly see that when m < n,* $\cos\left(\frac{m}{n}\pi\right)$ *and* $\cos\left(\frac{n-m}{n}\pi\right)$ *are opposites, and sum to zero. The only terms that will contribute any non-zero value to the sum are those where m = n. In each of those 2012 cases,* $\cos\left(\frac{m}{n}\pi\right)=\cos(1\pi)=-1$*, so the sum is* $(-1)\cdot 2012=\boxed{-2012}$.

2. $121_b=b^2+2b+1=324 \Rightarrow (b+1)^2=324 \Rightarrow b+1=18 \Rightarrow b=\boxed{17}$.

3. $y=(a^2+1)(x^2-1)-ax+7=(a^2+1)x^2-ax+(6-a^2)$. *Since the sum and product of roots are equal,* $\frac{a}{a^2+1}=\frac{6-a^2}{a^2+1} \Rightarrow a=6-a^2 \Rightarrow a^2+a-6=0 \Rightarrow (a+3)(a-2)=0 \Rightarrow a\in\{-3,2\}$. *Trying these values in the original equation results in the quadratics* $y=10x^2+3x-3$ *and* $y=5x^2-2x+2$. *Checking the discriminant of each shows that only* $y=10x^2+3x-3$ *has real roots, so the only acceptable value for a is* $\boxed{-3}$.

4. *Trying to simplify the complex fraction quickly turns messy. Instead, create an equation and start unraveling things:*

$$2+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z}}}=\frac{37}{13} \Rightarrow \cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z}}}=\frac{11}{13} \Rightarrow x+\cfrac{1}{y+\cfrac{1}{z}}=\frac{13}{11} \Rightarrow x=1,\ \cfrac{1}{y+\cfrac{1}{z}}=\frac{2}{11} \Rightarrow y+\frac{1}{z}=\frac{11}{2} \Rightarrow y=5,\ \frac{1}{z}=\frac{1}{2} \Rightarrow z=2.$$

So (x, y, z) = $\boxed{(1,5,2)}$.

5. *Assume the factored form is* $(x-r_1)(x-r_2)\dots(x-r_{17})$. *Each root must be selected from the values* ±17 *or* ±1*, with the product of those 17 roots being –17. Also, the sum of the roots is +7, which can only be created by starting with a 17, adding –1 ten times, and then using an equal quantity of –1's and +1's in the last six roots, to cancel each other out.*
In other words, the factorization is $(x-17)(x+1)^{10}(x+1)^3(x-1)^3=(x-17)(x+1)^{13}(x-1)^3$. *The x-term is formed by the sum of all possible products of 16 roots at a time (think about omitting a 17 once, a –1 thirteen times, and a +1 three times):* $(-1)^{13}(1)^3+13(17)(-1)^{12}(1)^3+3(17)(-1)^{13}(1)^2=-1+221-51=\boxed{169}$.

6. *We illustrate the general method using the arbitrary (not necessarily regular!) pentagon* $A_1A_2A_3A_4A_5$*, with interior angles labeled* $\alpha_1,\alpha_2,\alpha_3,\alpha_4$*, and* α_5 *and intersecting side extensions meeting at* S_1, S_2, S_3, S_4*, and* S_5 *(see Figure 6). Note that* $m\angle S_1=180°-(180°-\alpha_1)-(180°-\alpha_2)=\alpha_1+\alpha_2-180°$*, and so, similarly,* $m\angle S_2=\alpha_2+\alpha_3-180°$, $m\angle S_3=\alpha_3+\alpha_4-180°$*, and so on. Adding these five equations yields* $m\angle S_1+m\angle S_2+m\angle S_3+m\angle S_4+m\angle S_5=2(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)-5(180°)$.
Extending this idea to n sides, we have $m\angle S_1+m\angle S_2+\dots+m\angle S_n=2(180°(n-2))-n(180°)=\boxed{180n-720}$.