Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

____________ 1. There were 7 boys and 13 girls at a party. What percentage of the group were boys?

$x =$ ____________ 2. If $161 \equiv x \bmod 13$, and x is a single positive digit, determine the value of x.

____________ 3. Convert $r = 1.4727272...$ into a fraction expressed as the quotient of two relatively prime integers.

____________ 4. An integer, increased by the value of its cube, becomes 592,788.
What is the original integer?

Name: __________________ Team: __________________

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

35% **(Also accept "35")**

1. There were 7 boys and 13 girls at a party. What percentage of the group were boys?

$$\frac{7\ boys}{20\ total\ in\ group} = 35\%.$$

$x = \boxed{5}$

2. If $161 \equiv x \bmod 13$, and x is a single positive digit, determine the value of x.

Since $169 = 13 \cdot 13$, *we know that* 161 *is* 8 *short of a multiple of* 13 . . . *i.e.,* $x + 8 \equiv 13 \bmod 13$. *Subtracting* 8 *from both sides,* $x = 5$.

$\boxed{\frac{81}{55}}$

3. Convert $r = 1.4727272...$ into a fraction expressed as the quotient of two relatively prime integers.

Multiply both sides of the given equation by 10*, so that* $10r = 14 + x$ *where* $x = 0.727272\ldots$

Now note that $100x = 72 + x \Rightarrow x = \frac{8}{11}$. *Thus,* $10r = \frac{14 \cdot 11 + 8}{11} = \frac{162}{11} \Rightarrow r = \frac{81}{55}$.

$\boxed{84}$

4. An integer, increased by the value of its cube, becomes 592,788. What is the original integer?

Since $80 + 80^3 = 512{,}080$ *and* $90 + 90^3 = 729{,}090$, *the integer must be between* 80 *and* 90*, and closer to* 80*. Use modular arithmetic based on the last digit of* $x + x^3$:

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	7	4	5	6	3	2	9
$x+x^3$	0	2	0	0	$\boxed{8}$	0	2	0	0	$\boxed{8}$

---> *Only two values yield a last digit of 8.*

Thus our only two options are 84 *and* 89*, and* $\boxed{84}$ *is closer to* 80.

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$m\angle BDC =$ ________ 1. In *Figure 1*, $m\angle ABC = 90°$, and point D is located on $\overline{AC}$ such that $AB = BD = CD$. What is the measure of $\angle BDC$?

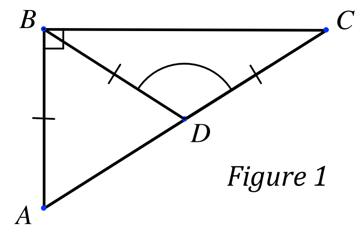


Figure 1

$m\angle SRQ =$ ________ 2. *Figure 2* shows an equilateral triangle sharing a common side with a square. Determine the measure of $\angle SRQ$.

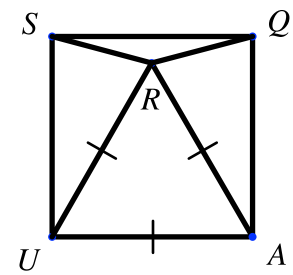


Figure 2

________ 3. In regular 9-sided polygon $A_1A_2 \ldots A_9$ *(Figure 3)*, what is the measure of the obtuse angle between diagonals A_1A_5 and A_4A_7?

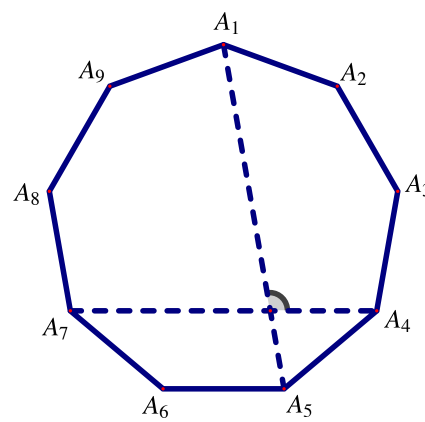


Figure 3

________ 4. The distances from a point to three vertices of a rectangle are 2, 5, and 10, as shown in *Figure 4*. What is the distance from the point to the fourth vertex?

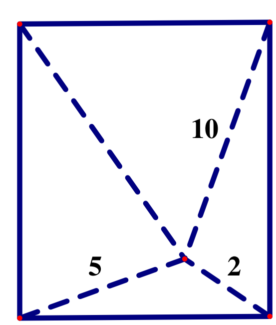


Figure 4

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event B

SOLUTIONS

$m\angle BDC =$ **120°**

1. In *Figure 1*, $m\angle ABC = 90°$, and point D is located on $\overline{AC}$ such that $AB = BD = CD$. What is the measure of $\angle BDC$?

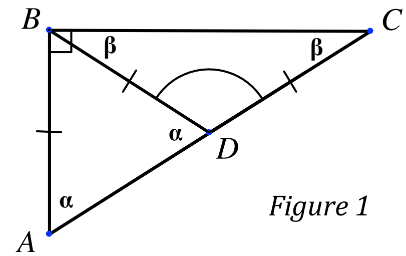


Figure 1

> *In* $\triangle ABC$, $\alpha + \beta + 90° = 180°$, *so* $\alpha + \beta = 90°$.
> *Also,* $m\angle BDC = 180° - \alpha = 180° - 2\beta$, *so* $\alpha = 2\beta$.
> *This means* $\beta = 30°$, *so* $m\angle BDC = 180° - 2(30°) = 120°$.

$m\angle SRQ =$ **150°**

2. *Figure 2* shows an equilateral triangle sharing a common side with a square. Determine the measure of $\angle SRQ$.

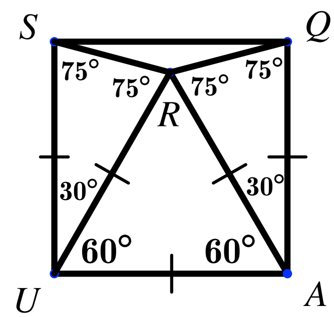


Figure 2

> *Note that* $\triangle SUR$ *and* $\triangle ARQ$ *are isosceles, allowing us to label the angles as shown in the figure. Focus on point R:*
> $m\angle SRQ + 75° + 75° + 60° = 360° \Rightarrow m\angle SRQ = 150°$.

100°

3. In regular 9-sided polygon $A_1A_2\ldots A_9$ *(Figure 3)*, what is the measure of the obtuse angle between diagonals A_1A_5 and A_4A_7?

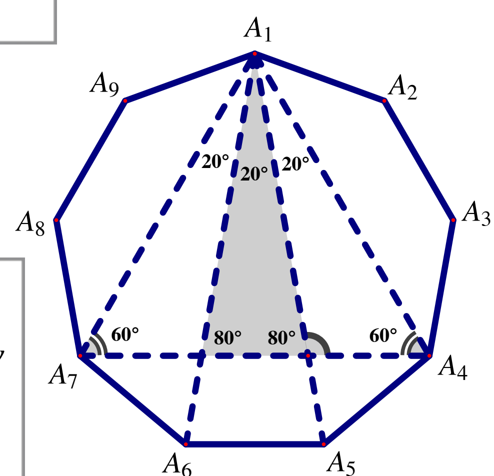


Figure 3

> $\triangle A_1A_4A_7$ *is equilateral, because each of its sides cut off an identical section of the nonagon. So* $m\angle A_7A_1A_4 = 60°$, *and by similar reasoning,* $m\angle A_7A_1A_6 = m\angle A_6A_1A_5 = m\angle A_5A_1A_4 = 20°$. *Using the shaded isosceles triangle, the desired obtuse angle has measure* $180° - 80° = 100°$.

11

4. The distances from a point to three vertices of a rectangle are 2, 5, and 10, as shown in *Figure 4*. What is the distance from the point to the fourth vertex?

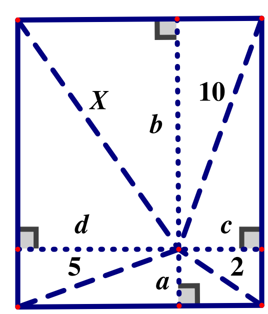


Figure 4

> *Through the interior point, draw line segments perpendicular to each side of the rectangle. Using Pythagorean relationships,*
>
> $$\begin{cases} a^2 + c^2 = 2^2 \\ b^2 + c^2 = 10^2 \\ b^2 + d^2 = X^2 \\ d^2 + a^2 = 5^2 \end{cases} \Rightarrow a^2 + b^2 + c^2 + d^2 = 4 + X^2 = 100 + 25$$
>
> $$X^2 = 121$$
>
> $$X = 11$$

Note the theorem proven here: The sum of the squares of opposing distances from the interior point must be equal!

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Determine exactly the value of $\cos\frac{\pi}{1}+\cos\frac{\pi}{2}+\cos\frac{\pi}{3}$.

$m\angle A=$ ________________ 2. Find the radian measure of the smallest positive angle A for which $\cos A$ has the same value as $\sin 210°$.

________________ 3. In the *xy*-plane, the graphs of $y=\sin x$ and $y=\frac{1}{2}$ intersect at points P and Q, where P is in the first quadrant and Q is in the second quadrant. In terms of π, determine exactly the smallest possible distance between P and Q.

$\cos\angle DAQ=$ ________________ 4. In rectangle *ABCD (Figure 4)*, $AB=8$ and $AD=12$. ΔAPQ is a right triangle, as shown. If $\cos\angle CPQ=\frac{4}{5}$, determine exactly the value of $\cos\angle DAQ$.

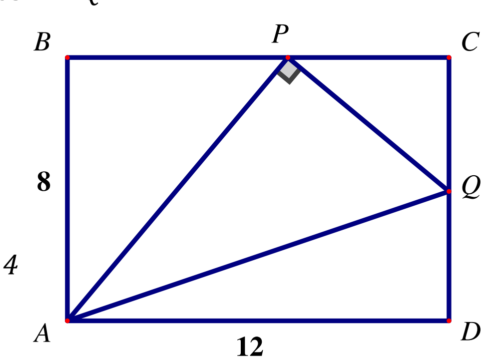


Figure 4

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event C

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{-\frac{1}{2}}$

1. Determine exactly the value of $\cos\frac{\pi}{1}+\cos\frac{\pi}{2}+\cos\frac{\pi}{3}$.

$$\cos\pi+\cos\frac{\pi}{2}+\cos\frac{\pi}{3}=(-1)+0+\frac{1}{2}=-\frac{1}{2}.$$

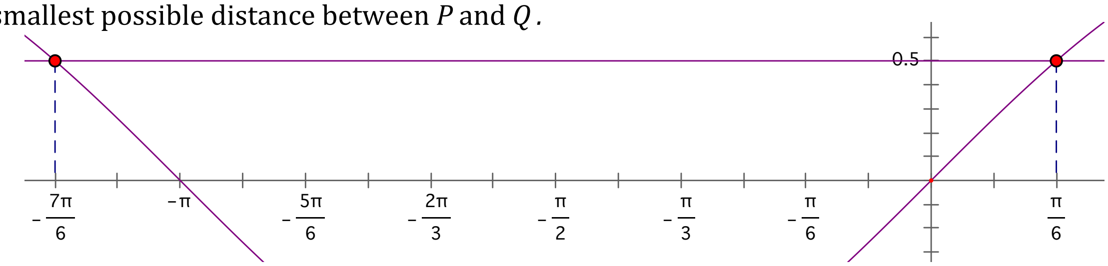
$m\angle A=\boxed{\frac{2\pi}{3}}$

2. Find the radian measure of the smallest positive angle A for which $\cos A$ has the same value as $\sin 210°$.

$\sin 210°=-\sin 30°=-\frac{1}{2}$. *The earliest that cosine takes on negative values is in quadrant II, so we seek* $\frac{\pi}{2}<A<\pi$ *for which* $\cos A=-\frac{1}{2}$. *Since* $\cos\frac{\pi}{3}=\frac{1}{2}$, *reflect across the y-axis to* $A=\frac{2\pi}{3}$.

$\boxed{\frac{4\pi}{3}}$

3. In the *xy*-plane, the graphs of $y=\sin x$ and $y=\frac{1}{2}$ intersect at points *P* and *Q*, where *P* is in the first quadrant and *Q* is in the second quadrant. In terms of π, determine exactly the smallest possible distance between *P* and *Q*.



$\cos\angle DAQ=$

$\boxed{\frac{24}{25}}$

4. In rectangle *ABCD (Figure 4)*, *AB* = 8 and *AD* = 12. $\triangle APQ$ is a right triangle, as shown. If $\cos\angle CPQ=\frac{4}{5}$, determine exactly the value of $\cos\angle DAQ$.

Angle chasing shows us that $\angle BAP\cong\angle CPQ$, *so*

$\cos\angle BAP=\frac{4}{5}=\frac{8}{AP}$, *and AP* = 10. *Thus* $\triangle ABP$ *is 6-8-10.*

This means PC = 6, *so* $\cos\angle CPQ=\frac{4}{5}=\frac{6}{PQ}\Rightarrow PQ=\frac{15}{2}$.

Considering AP's length as $\frac{20}{2}$, $\triangle APQ$ is $\frac{15}{2},\frac{20}{2},\frac{25}{2}$.

$\cos\angle DAQ=\frac{12}{25/2}=12\cdot\frac{2}{25}=\frac{24}{25}$.

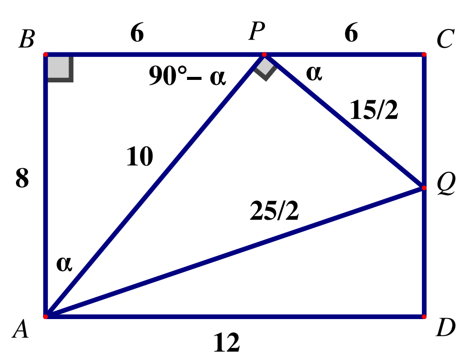


Figure 4

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

__________________ 1. Write, in $x^2 + bx + c = 0$ form, the quadratic equation whose roots are −3 and 1.

__________________ 2. Find the remainder when $x^{13} + 1$ is divided by $x - 1$.

__________________ 3. Calculate the difference between the two roots of the equation $x^2 - px + \frac{p^2 - 1}{4} = 0$.

p = __________________ 4. If the solutions (for x) of $x^2 + px + q = 0$ are the cubes of the solutions for $x^2 + mx + n = 0$,

q = __________________ express p and q as polynomials in terms of m and n.

Name: __________________________ Team: __________________________

Minnesota State High School Mathematics League

2011-12 Meet 1, Individual Event D

SOLUTIONS

$\boxed{x^2+2x-3=0}$

1. Write, in $x^2+bx+c=0$ form, the quadratic equation whose roots are –3 and 1.

$(x+3)(x-1)=x^2-1x+3x-3=x^2+2x-3=0$.

$\boxed{2}$

2. Find the remainder when $x^{13}+1$ is divided by x – 1.

By the Remainder Theorem, if $f(x)=x^{13}+1$, *then the remainder when f(x) is divided by x – 1 is simply* $f(1)=(1)^{13}+1=1+1=2$.

$\boxed{1}$

(Also accept "–1")

3. Calculate the difference between the two roots of the equation $x^2-px+\dfrac{p^2-1}{4}=0$.

[Mathematics Teacher]

$\dfrac{p^2-1}{4}$ *is the product of the two roots, and suspiciously factors into* $\dfrac{p-1}{2}$ *and* $\dfrac{p+1}{2}$.

Check the sum of the roots: $\dfrac{p-1}{2}+\dfrac{p+1}{2}=\dfrac{2p}{2}=p$, *which checks vs. the coefficient of the x term.*

So the two roots are $\dfrac{p-1}{2}$ *and* $\dfrac{p+1}{2}$, *and their difference is* $\dfrac{p+1}{2}-\dfrac{p-1}{2}=\dfrac{2}{2}=1$.

$p = \boxed{m^3-3mn}$

$q = \boxed{n^3}$

Graders:
1 point for each correct value

4. If the solutions (for *x*) of $x^2+px+q=0$ are the cubes of the solutions for $x^2+mx+n=0$, express *p* and *q* as polynomials in terms of *m* and *n*.

[Mathematics Teacher]

Let the two roots of $x^2+mx+n=0$ *be* R_1 *and* R_2. *Using sum and product of roots,* $R_1+R_2=-m$, *and* $R_1R_2=n$. *Also, the roots of* $x^2+px+q=0$ *are* $(R_1)^3$ *and* $(R_2)^3$, *so* $(R_1)^3+(R_2)^3=-p$, *and* $(R_1)^3(R_2)^3=q$. *So we have* $\boxed{q=n^3}$. *Factoring:* $-p=(R_1)^3+(R_2)^3=(R_1+R_2)\left((R_1)^2-R_1R_2+(R_2)^2\right)$

$=(-m)\left((R_1+R_2)^2-3R_1R_2\right)=(-m)(m^2-3n)=-m^3+3mn$, *so* $\boxed{p=m^3-3mn}$.

Minnesota State High School Mathematics League

2011-12 Meet 1, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

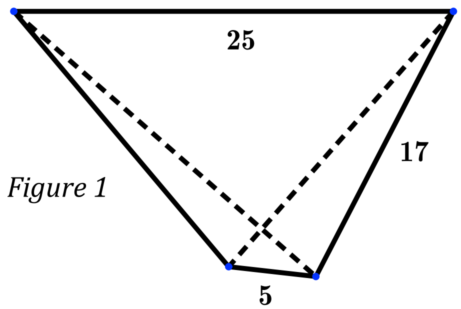


Figure 1

______________ 1. The diagonals of the quadrilateral shown in *Figure 1* are perpendicular. Lengths of three of the quadrilateral's sides are 5, 17, and 25. What is the length of the fourth side?

______________ 2. A polynomial has a remainder of 3 when divided by $(x - 1)$ and a remainder of 5 when divided by $(x - 3)$. What is the remainder when the polynomial is divided by $(x - 1)(x - 3)$?

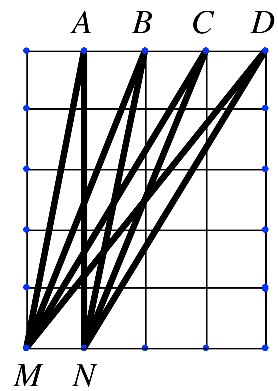


Figure 3

______________ 3. In *Figure 3*, each small grid square measures 1 square unit. Determine exactly the sum of the measures of angles *MAN, MBN, MCN,* and *MDN.*

______________ 4. A store advertised a one-day sale during which it would be selling appliances for 50% off the regular price. In addition, customers using the store's credit card could, after the price reduction, still get their regular 10% discount. A mathematically-ignorant salesman sold items to these customers at 60% off the regular price. If he accounted for $15,000 of store credit card sales, how much did his error cost the store in losses?

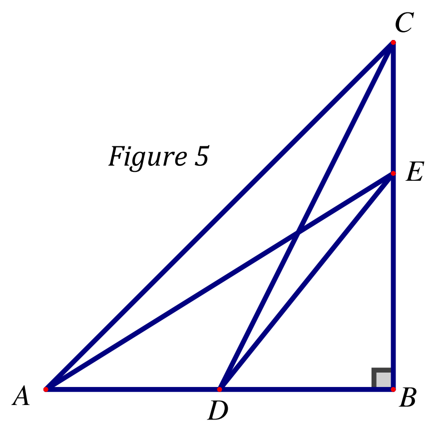


Figure 5

______________ 5. In right triangle *ABC (Figure 5)*, $AB = BC$.
If $\tan\angle BDE = 2\tan\angle BAE = 3\tan\angle BCD$,
determine exactly the value of $\tan\angle CAE + \tan\angle ACD$.

______________ 6. List all integers $n > 3$ such that $n - 3$ divides evenly into $n^2 - n$.

Team: ______________________

Minnesota State High School Mathematics League

2011-12 Meet 1, Team Event

SOLUTIONS (page 1)

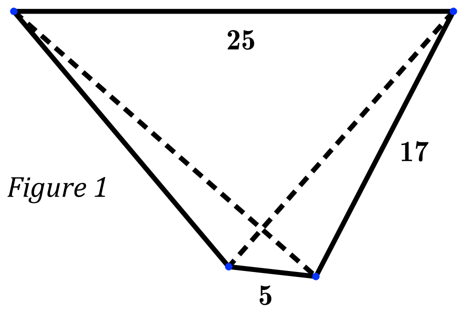


Figure 1

Answer: 19

1. The diagonals of the quadrilateral shown in *Figure 1* are perpendicular. Lengths of three of the quadrilateral's sides are 5, 17, and 25. What is the length of the fourth side?

Answer: $x+2$

2. A polynomial has a remainder of 3 when divided by $(x - 1)$ and a remainder of 5 when divided by $(x - 3)$. What is the remainder when the polynomial is divided by $(x - 1)(x - 3)$?

[Mathematics Teacher]

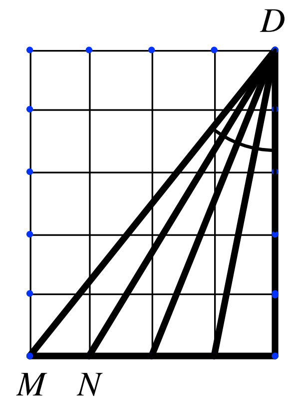


Figure 3

Answer: $\tan^{-1}\left(\frac{4}{5}\right)$

(Also accept any equivalent inverse trig expressions)

3. In *Figure 3*, each small grid square measures 1 square unit. Determine exactly the sum of the measures of angles *MAN, MBN, MCN,* and *MDN.*

Answer: \$1875

4. A store advertised a one-day sale during which it would be selling appliances for 50% off the regular price. In addition, customers using the store's credit card could, after the price reduction, still get their regular 10% discount. A mathematically-ignorant salesman sold items to these customers at 60% off the regular price. If he accounted for \$15,000 of store credit card sales, how much did his error cost the store in losses?

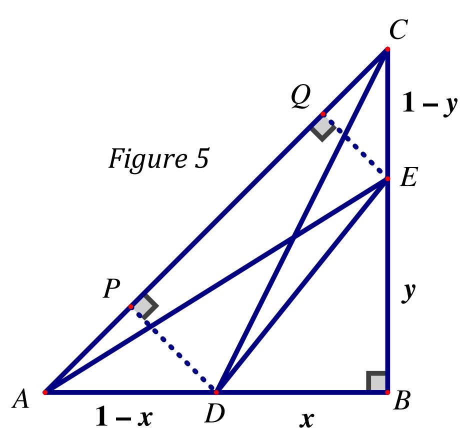


Figure 5

Answer: $\frac{10}{21}$

5. In right triangle *ABC (Figure 5)*, $AB = BC$.
If $\tan\angle BDE = 2\tan\angle BAE = 3\tan\angle BCD$,
determine exactly the value of $\tan\angle CAE + \tan\angle ACD$.

Answer: 4, 5, 6, 9

Graders: 1 point for each correct value

6. List all integers $n > 3$ such that $n - 3$ divides evenly into $n^2 - n$.

[Mathematics Teacher]

Minnesota State High School Mathematics League

2011-12 Meet 1, Team Event

SOLUTIONS (page 2)

1. *This is like an "inside-out" version of the rectangle from Event B, problem #4. If you were wise to check the Event B solutions page before entering the Team Event, you would have noticed a useful theorem that applies here:*

$$5^2+25^2=17^2+X^2 \Rightarrow 25+625=289+X^2 \Rightarrow X^2=361 \Rightarrow X=\boxed{19}.$$

2. *Let the original polynomial be P(x). Express* $P(x)=Q(x)\cdot(x-1)(x-3)+R(x)$*, where Q(x) is the quotient polynomial and R(x) is the remainder polynomial we're looking for. Since R(x) is the remainder after dividing by a quadratic, R(x) can be no more than a linear polynomial. Set* $R(x)=ax+b$*. P(1) = 3, so a + b = 3. Also, P(3) = 5, so 3a + b = 5. By elimination,* $2a=2 \Rightarrow a=1 \Rightarrow b=2$*, and* $R(x)=\boxed{x+2}$*.*

3. *Translate triangles MAN, MBN, and MCN so that the images of points A, B, and C align with point D. The sum of the four angles can now be seen as a single large acute angle at point D, inside a right triangle with legs of lengths 4 and 5.*

The easiest way to express this angle exactly is to use the inverse tangent function: $\boxed{\tan^{-1}\left(\frac{4}{5}\right)}$

4. *The salesman took in 40% of the regular price p. In other words,* $.40p=15000 \Rightarrow p=\$37{,}500$*. He* ***should*** *have first taken 50% off:* $.50(\$37{,}500)=\$18{,}500$*, and* ***then*** *taken another 10% off, leaving 90%:* $.90(\$18{,}500)=\$16{,}875$*.*

Therefore, his error cost the store \$16,875 – \$15,000 = $\boxed{\$1875}$*.*

5. *Let AB = BC = 1, and set BD = x , BE = y. Dropping perpendiculars from D and E to* $\overline{AC}$ *creates 45-45-90 triangles ADP and CEQ. AD = 1 – x, so AP = DP =* $\frac{1-x}{\sqrt{2}}$ *and CP =* $\sqrt{2}-AP=\sqrt{2}-\frac{1-x}{\sqrt{2}}=\frac{2-(1-x)}{\sqrt{2}}=\frac{1+x}{\sqrt{2}}$*. Thus,* $\tan\angle ACD=\frac{DP}{CP}=\frac{1-x}{1+x}$*.*

By similar reasoning, $\tan\angle CAE=\frac{EQ}{AQ}=\frac{1-y}{1+y}$*. Using the given fact that* $\tan\angle BDE=2\tan\angle BAE=3\tan\angle BCD$*, we have*

$$\frac{y}{x}=2y=3x \Rightarrow x=\frac{1}{2},\ y=\frac{3}{4}, \text{ and so } \tan\angle ACD+\tan\angle CAE=\frac{1-x}{1+x}+\frac{1-y}{1+y}=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}+\frac{1-\frac{3}{4}}{1+\frac{3}{4}}=\frac{\frac{1}{2}}{\frac{3}{2}}+\frac{\frac{1}{4}}{\frac{7}{4}}=\frac{1}{3}+\frac{1}{7}=\boxed{\frac{10}{21}}.$$

6. *Synthetically divide:*

$$\begin{array}{c|ccc} 3 & 1 & -1 & 0 \\ & 1 & 2 & \boxed{6} \end{array}$$

So $\frac{n^2-n}{n-3}=n+2+\frac{6}{n-3}$*. We need n – 3 to evenly divide 6 so that* $\frac{6}{n-3}$ *becomes part of the quotient. 6 has four divisors: {1, 2, 3, 6}, so set* $n-3=\{1,2,3,6\}$ *so that* $n\in\boxed{\{4,5,6,9\}}$*.*