Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Determine exactly the value of $\frac{8^4}{4^8}$.

$x =$ ________________ 2. Determine exactly all solutions to $\frac{2x+1}{x+3} - \frac{5x+4}{4x+12} = 1$.

________________ 3. How many positive integers satisfy $n^{2/3} < 64 < n^{3/4}$?

________________ 4. Find two distinct 2-digit prime divisors of $5^{18} - 7^{12}$.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{\frac{1}{16}}$

1. Determine exactly the value of $\frac{8^4}{4^8}$.

$$\frac{8^4}{4^8} = \frac{\left(2^3\right)^4}{\left(2^2\right)^8} = \frac{2^{12}}{2^{16}} = \frac{1}{2^4} = \frac{1}{16}.$$

$x = \boxed{-12}$

2. Determine exactly all solutions to $\frac{2x+1}{x+3} - \frac{5x+4}{4x+12} = 1$.

Noting that $x \neq -3$, *multiply the top and bottom of the first fraction by 4.*

Then $\frac{8x+4}{4x+12} - \frac{5x+4}{4x+12} = \frac{3x}{4x+12} = 1$, *so* $3x = 4x + 12 \Rightarrow x = -12$.

$\boxed{255}$

3. How many positive integers satisfy $n^{2/3} < 64 < n^{3/4}$?

Isolating the left inequality, $64 = 512^{2/3}$, *so* $n^{2/3} < 512^{2/3} \Rightarrow n < 512$.

Similarly with the right inequality: $64 = 256^{3/4}$, *so* $256^{3/4} < n^{3/4} \Rightarrow 256 < n$.

Since 256 < *n* < 512, there are 511 – 257 + 1 = 255 positive integral values that work.

$\boxed{19 \text{ and } 29}$

(Graders: award 1 point for each correct divisor)

4. Find two distinct 2-digit prime divisors of $5^{18} - 7^{12}$.

Let $x = 5^3$, $y = 7^2$. *Then*

$$5^{18} - 7^{12} = x^6 - y^6 = \left(x^3 - y^3\right)\left(x^3 + y^3\right) = \left(x - y\right)\left(x^2 + xy + y^2\right)\left(x + y\right)\left(x^2 - xy + y^2\right).$$

Searching for 2-digit divisors, $x - y = 5^3 - 7^2 = 125 - 49 = 76 = 4 \cdot 19$, *so one of the divisors is* 19.

Also, $x + y = 5^3 + 7^2 = 125 + 49 = 174 = 2 \cdot 3 \cdot 29$, *so 29 is another 2-digit divisor.*

(As it so happens, $x^2 + xy + y^2$ *and* $x^2 - xy + y^2$ *only generate the factors 2, 3, 3967, and 24151.)*

Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

____________ 1. Three concentric circles have radii of lengths 5, 12, and 13. What is the length of the shortest line segment on which lies a point from each of the three circles?

$m\overset{\frown}{OX}$ = ____________ 2. *Figure 2* shows circle *O* with tangent $\overline{PT}$ and secant $\overline{PB}$ (which intersects *O* at *A* and *B*). A semicircle is drawn with $\overline{AO}$ as diameter, intersecting $\overline{OT}$ at *X*. If $m\angle TPB = 40°$, calculate the degree measure of $\overset{\frown}{OX}$.

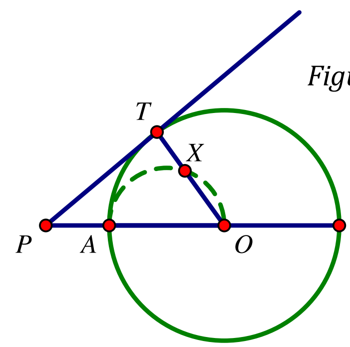


Figure 2

____________ 3. In *Figure 3*, the two medium-sized circles each have a diameter of length 4, while the small circle has a diameter of length 2. Areas *I, II, III,* and *IV* do not include any areas where circles overlap. What must be the length of the large circle's diameter in order for the sum of regions *I* + *II* + *III* to equal the area of region *IV*?

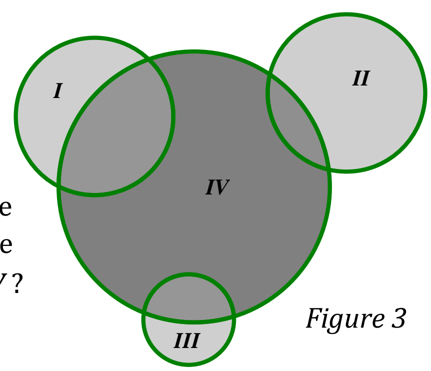


Figure 3

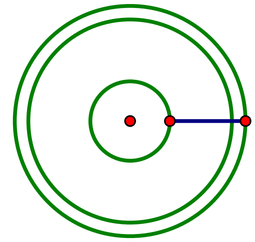
____________ 4. In creating a exhibit on the growth of American transportation, an artist begins with a 1-foot-radius tricycle tire and a 9-foot-radius tractor tire. The artist sets the two tires on the ground, treads facing each other, on a painted line. If the entire exhibit must fit into a space no more than 22 feet wide, what is the radius of the largest tire that the artist can set on the ground between the tricycle and tractor tires?

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event B

SOLUTIONS



8

[1st High School Mathematics League Problem Book]

1. Three concentric circles have radii of lengths 5, 12, and 13. What is the length of the shortest line segment on which lies a point from each of the three circles?

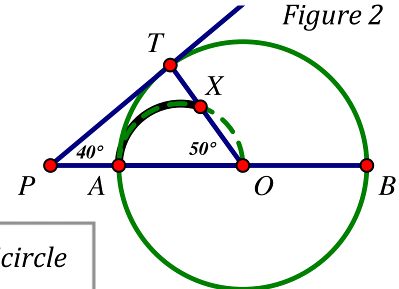
The shortest segment's length = the largest radius minus the smallest radius, or 13 – 5 = 8.

$m\,\overset{\frown}{OX}$ = **80°**

[New York City Contest Problem Book]

2. *Figure 2* shows circle *O* with tangent $\overline{PT}$ and secant $\overline{PB}$ (which intersects *O* at *A* and *B*). A semicircle is drawn with $\overline{AO}$ as diameter, intersecting $\overline{OT}$ at *X*. If $m\angle TPB = 40°$, calculate the degree measure of $\overset{\frown}{OX}$.

Figure 2



In right $\triangle PTO$, $m\angle AOX = 50°$. *This inscribed angle of the semicircle subtends* $\overset{\frown}{AX}$, *so* $m\overset{\frown}{AX} = 2 \cdot m\angle AOX = 100°$ *and* $m\overset{\frown}{OX} = 80°$.

6

[Mathematics Teacher, Feb. 2006]

3. In *Figure 3*, the two medium-sized circles each have a diameter of length 4, while the small circle has a diameter of length 2. Areas *I, II, III,* and *IV* do not include any areas where circles overlap. What must be the length of the large circle's diameter in order for the sum of regions *I + II + III* to equal the area of region *IV*?

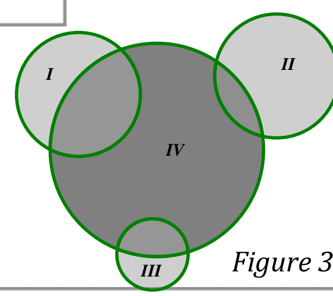


Figure 3

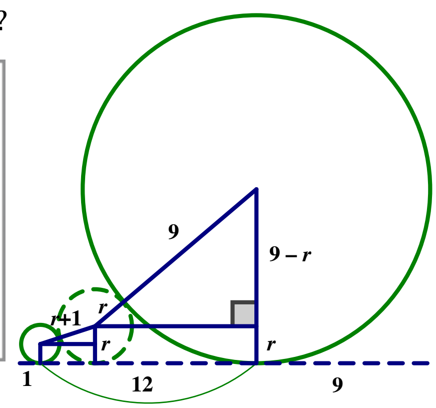
Let X = total area of the 3 regions of overlap. Then I + II + III + X = $\pi(2)^2 + \pi(2)^2 + \pi(1)^2 = 9\pi$, *and IV + X =* πr^2, *where r is the length of the large circle's radius. Substituting based on X yields I + II + III +* πr^2 *– IV =* 9π, *but since I + II + III = IV, we have* $\pi r^2 = 9\pi \;\Rightarrow\; r = 3 \;\Rightarrow\; d = 6$.

$\frac{9}{4}$ (*feet*) **or** 2.25 (*feet*) **or** 2'3"

4. In creating a exhibit on the growth of American transportation, an artist begins with a 1-foot-radius tricycle tire and a 9-foot-radius tractor tire. The artist sets the two tires on the ground, treads facing each other, on a painted line. If the entire exhibit must fit into a space no more than 22 feet wide, what is the radius of the largest tire that the artist can set on the ground between the tricycle and tractor tires?

Let r = radial length of the new tire. The small right triangle in the diagram has a base of length $\sqrt{(r+1)^2 - (r-1)^2} = \sqrt{4r} = 2\sqrt{r}$, *while the large right triangle has base* $\sqrt{(9+r)^2 - (9-r)^2} = \sqrt{36r} = 6\sqrt{r}$.

So $2\sqrt{r} + 6\sqrt{r} = 12 \;\Rightarrow\; 8\sqrt{r} = 12 \;\Rightarrow\; \sqrt{r} = \frac{3}{2}$, *and* $r = \frac{9}{4}$.



Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

____________ 1. The Reduce feature of a copy machine is set so that it shrinks an image to 80% of its original area. If a square of area 1 is reduced, then that result reduced, and so on until ten copies have been made, calculate the sum of the areas of the original square and its copies.

____________ 2. Peter Plod used his calculator to find the sum of 1 + 3 + 5 + ... + 201. The monotony got the best of him, however, and he hit the minus key rather than plus key prior to one of the numbers. As a result, he got the (incorrect) total of 10023. Which number did Peter accidentally subtract?

$f(15)=$ ____________ 3. Let f be a function with the property that for all whole numbers n,

$$f(0)=a^2b \qquad f(1)=\frac{1}{ab} \qquad f(n+1)=f(n)\cdot f(n-1) \text{ for } n \geq 1$$

In terms of a and b, determine exactly an expression for $f(15)$.

____________ 4. If $P(n)=\dfrac{1}{n^2-1}-\dfrac{1}{n^3-n}$, determine exactly the value of $P(2)+P(3)+P(4)+\ldots+P(2011)$.

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event C

SOLUTIONS

$\boxed{\approx 4.571}$ **or** $\boxed{\approx 4.570}$

1. The Reduce feature of a copy machine is set so that it shrinks an image to 80% of its original area. If a square of area 1 is reduced, then that result reduced, and so on until ten copies have been made, calculate the sum of the areas of the original square and its copies.

This is a geometric series, with $a_1 = 1$ *and* $r = 0.8$. *We seek* $S_{11} = \dfrac{a_1\left(r^{11}-1\right)}{r-1} = \dfrac{0.8^{11}-1}{0.8-1} \approx 4.571$.

$\boxed{89}$

2. Peter Plod used his calculator to find the sum of 1 + 3 + 5 + ... + 201. The monotony got the best of him, however, and he hit the minus key rather than plus key prior to one of the numbers. As a result, he got the (incorrect) total of 10023. Which number did Peter accidentally subtract?

Since $1 + 3 + \ldots + (2n-1) = n^2$, $1 + 3 + \ldots + 201 = 1 + 3 + \ldots + \left(2\cdot 101 - 1\right) = 101^2 = 10201$.
Incorrectly subtracting a number x will actually create an error of 2*x, because x was subtracted and also not added. So* $2x = 10201 - 10023 = 178$, and $x = 89$.

$f(15) = \boxed{\dfrac{a^{144}}{b^{233}}}$

3. Let f be a function with the property that for all whole numbers n,

$$f(0) = a^2 b \qquad f(1) = \frac{1}{ab} \qquad f(n+1) = f(n)\cdot f(n-1) \text{ for } n \ge 1$$

In terms of a and b, determine exactly an expression for $f(15)$.

$f(2) = f(1)\cdot f(0) = \dfrac{1}{ab}\cdot a^2 b = a$, $f(3) = f(2)\cdot f(1) = a\cdot\dfrac{1}{ab} = \dfrac{1}{b}$, $f(4) = f(3)\cdot f(2) = \dfrac{1}{b}\cdot a = \dfrac{a}{b}$

$f(5) = f(4)\cdot f(3) = \dfrac{a}{b}\cdot\dfrac{1}{b} = \dfrac{a}{b^2}$, $f(6) = f(5)\cdot f(4) = \dfrac{a}{b^2}\cdot\dfrac{a}{b} = \dfrac{a^2}{b^3}$ ... *and we start to see that the powers of a and b are following the Fibonacci sequence.* $f(15) = \dfrac{a^{F_{12}}}{b^{F_{13}}} = \dfrac{a^{144}}{b^{233}}$.

$\boxed{\dfrac{1005}{2012}}$

4. If $P(n) = \dfrac{1}{n^2-1} - \dfrac{1}{n^3-n}$, determine exactly the value of $P(2) + P(3) + P(4) + \ldots + P(2011)$.

$P(n) = \dfrac{1}{n^2-1} - \dfrac{1}{n^3-n} = \dfrac{1}{n^2-1} - \dfrac{1}{n\left(n^2-1\right)} = \dfrac{n-1}{n(n-1)(n+1)} = \dfrac{1}{n(n+1)}$, *so* $P(2) = \dfrac{1}{6} = \dfrac{1}{2} - \dfrac{1}{3}$,

$P(3) = \dfrac{1}{12} = \dfrac{1}{3} - \dfrac{1}{4}$, $P(4) = \dfrac{1}{20} = \dfrac{1}{4} - \dfrac{1}{5}$, *etc. Thus* P(2) + P(3) + P(4) + ... + P(2011) =

$\left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \left(\dfrac{1}{4} - \dfrac{1}{5}\right) + \ldots + \left(\dfrac{1}{2011} - \dfrac{1}{2012}\right) = \dfrac{1}{2} - \dfrac{1}{2012} = \dfrac{1005}{2012}$.

Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

All problems on this event make use of the points A(2,0) and B(0,3).

______________________ 1. A hyperbola has two asymptotes which intersect at *A*. If one of the asymptotes is $\overleftrightarrow{AB}$, write the equation of the other asymptote.

______________________ 2. Write the equation of the circle which has $\overline{AB}$ as a diameter.

______________________ 3. There are two parabolas, each of which has its vertex at *A* and passes through *B*. Write the equation of each parabola. ***(1 point each)***

____

______________________ 4. Write the equation of the ellipse, centered at *A*, which passes through *B* and has a focus at the origin.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2010-11 Meet 4, Individual Event D

SOLUTIONS

All problems on this event make use of the points A(2,0) and B(0,3).

$y=\frac{3}{2}(x-2)$

(or equivalent)

1. A hyperbola has two asymptotes which intersect at A. If one of the asymptotes is $\overleftrightarrow{AB}$, write the equation of the other asymptote.

The slope of $\overleftrightarrow{AB}$ *is* $-\frac{3}{2}$*, so the slope of the other asymptote must be* $+\frac{3}{2}$*. Using point-slope form along with the coordinates of point A, the equation of the other asymptote is* $y-0=\frac{3}{2}(x-2)$.

$(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{4}$

(or equivalent)

2. Write the equation of the circle which has $\overline{AB}$ as a diameter.

$\left(1,\frac{3}{2}\right)=MP_{\overline{AB}}$ *= center of the circle, while* $\sqrt{2^2+3^2}=\sqrt{13}=AB=2r$.

Using standard form for a circle: $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{\sqrt{13}}{2}\right)^2=\frac{13}{4}$.

$y=\frac{3}{4}(x-2)^2$

$x=-\frac{2}{9}y^2+2$

(or equivalents)

3. There are two parabolas, each of which has its vertex at A and passes through B. Write the equation of each parabola. ***(1 point each)***

One parabola is vertically oriented, while the other is horizontally oriented:

$y=a(x-2)^2$	$x-2=a(y-0)^2$
$3=a(-2)^2 \Rightarrow a=\frac{3}{4}$	$0-2=a(3)^2 \Rightarrow a=-\frac{2}{9}$
$y=\frac{3}{4}(x-2)^2$	$x=-\frac{2}{9}y^2+2$

$\frac{(x-2)^2}{16}+\frac{y^2}{12}=1$

(or equivalent)

4. Write the equation of the ellipse, centered at A, which passes through B and has a focus at the origin.

This problem can be solved by setting up a system of equations in a^2 *and* b^2*, but it can be rendered almost trivial by simply using the definition of the conic. In the case of an ellipse, it is the locus of points, the sum of whose distances to each of two foci is constant. If one focus is at (0,0) and the center is at (2,0), the other focus must be at (4,0). Point B is 3 units from one focus and 5 units from the other, for a sum of 8. This sum is also the length of the major axis (2a), so a = 4.*

Using $c^2=a^2-b^2$ *with c = 2 yields* $b^2=12$*, so the equation of the ellipse is* $\frac{(x-2)^2}{16}+\frac{y^2}{12}=1$.

Minnesota State High School Mathematics League

2010-11 Meet 4, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

________________ 1. a, b, and n are all positive integers. The expansion of $(a+b)^n$, written in order of decreasing powers of a, includes three consecutive terms equal to 67500, 30375, and 7290. What would the term immediately preceding 67500 equal?

$n =$ ________________ 2. A circle has inscribed in it a triangle, a regular 21-gon, a regular 28-gon, and a regular n-gon, all overlaid on top of each other. The triangle shares one side with the 21-gon, another side with the 28-gon, and the third side with the n-gon. What is the largest possible value of n?

________________ 3. Given $A(2, 0)$ and $B(0, 3)$, there are two circles which are tangent to $\overleftrightarrow{AB}$ and both axes. Determine exactly the difference in length of the circles' radii.

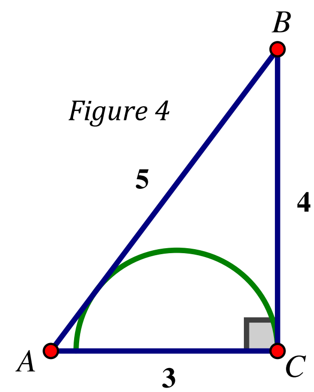


Figure 4

________________ 4. In $\triangle ABC$ *(Figure 4)*, $AC = 3$, $BC = 4$, and $AB = 5$. A semicircle, with its center on $\overline{AC}$, is tangent to both $\overline{AB}$ and $\overline{BC}$. Determine exactly the length of the semicircle's radius.

________________ 5. An algorithmically-challenged painter is hired to paint a 200-foot-long line in the center of a straight road. Never thinking to move the can as he works, his unfortunate procedure is:

(1) Fill his brush, paint one foot, and return one foot to the can;

(2) Fill his brush, walk one foot to the end of the painted line, paint one foot, and return two feet to the can;

(3) Fill his brush, walk two feet to the end of the painted line, paint one foot, and return three feet to the can ... and so on.

After one hour, the painter is back to the can, having painted 100 feet. After the second hour, he is back to the can, having painted an additional 50 feet. Assuming that he fills his brush, walks, and paints at constant rates, how much additional time will it take him to be back at the can with all 200 feet of line painted?

________________ 6. $f(x)$ and $g(x)$ are straight lines whose x-intercepts and y-intercepts are reflections of each other across the origin. If $f(g(x)) = 9x + 12$, list all possible values for $f(10)$.

Team: ____________________________

Minnesota State High School Mathematics League

2010-11 Meet 4, Team Event

SOLUTIONS (page 1)

Answer: 84375

1. a, b, and n are all positive integers. The expansion of $\left(a+b\right)^n$, written in order of decreasing powers of a, includes three consecutive terms equal to 67500, 30375, and 7290. What would the term immediately preceding 67500 equal?

Answer: n = 84

2. A circle has inscribed in it a triangle, a regular 21-gon, a regular 28-gon, and a regular n-gon, all overlaid on top of each other. The triangle shares one side with the 21-gon, another side with the 28-gon, and the third side with the n-gon. What is the largest possible value of n?

[1st High School Mathematics League Problem Book]

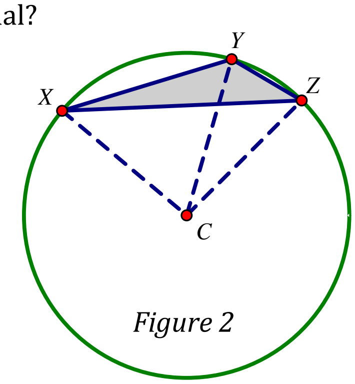


Figure 2

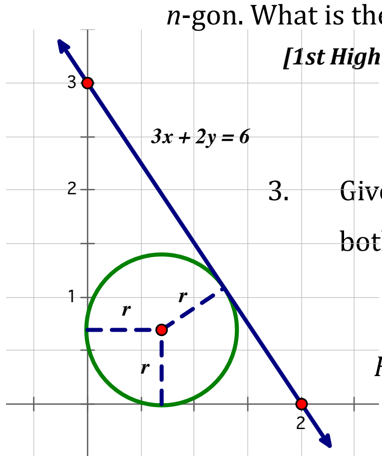


Figure 3

Answer: $\sqrt{13}$

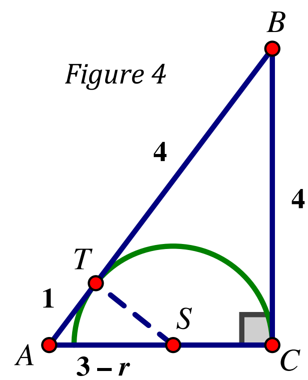
3. Given $A(2, 0)$ and $B(0, 3)$, there are two circles which are tangent to $\overleftrightarrow{AB}$ and both axes. Determine exactly the difference in length of the circles' radii.

Answer: $\frac{4}{3}$

4. In $\triangle ABC$ *(Figure 4)*, $AC = 3$, $BC = 4$, and $AB = 5$. A semicircle, with its center on $\overline{AC}$, is tangent to both $\overline{AB}$ and $\overline{BC}$. Determine exactly the length of the semicircle's radius.

[New York City Contest Problem Book]

Figure 4



Answer: 1 hour, 20 mins.

5. An algorithmically-challenged painter is hired to paint a 200-foot-long line in the center of a straight road. Never thinking to move the can as he works, his unfortunate procedure is:
   (1) Fill his brush, paint one foot, and return one foot to the can;
   (2) Fill his brush, walk one foot to the end of the painted line, paint one foot, and return two feet to the can;
   (3) Fill his brush, walk two feet to the end of the painted line, paint one foot, and return three feet to the can ... and so on.

   After one hour, the painter is back to the can, having painted 100 feet. After the second hour, he is back to the can, having painted an additional 50 feet. Assuming that he fills his brush, walks, and paints at constant rates, how much additional time will it take him to be back at the can with all 200 feet of line painted?

Answer: –27, 24

6. *f(x)* and *g(x)* are straight lines whose x-intercepts and y-intercepts are reflections of each other across the origin. If $f(g(x)) = 9x + 12$, list all possible values for $f(10)$.

Minnesota State High School Mathematics League

2010-11 Meet 4, Team Event

SOLUTIONS (page 2)

1. *Note that* $67500 = 2^2 \cdot 3^3 \cdot 5^4$, $30375 = 3^5 \cdot 5^3$, *and* $7290 = 2 \cdot 3^6 \cdot 5$. *The powers of 3 are strictly increasing, while the powers of 5 are strictly decreasing. Therefore, let a = 5, b = 3. Re-express* 67500, 30375, and 7290 as $2^2 \cdot 5 \cdot \left(3^3 \cdot 5^3\right)$, $3 \cdot 5 \cdot \left(3^4 \cdot 5^2\right)$, and $2 \cdot 3 \cdot \left(3^5 \cdot 5^1\right)$. *The coefficients are* 20, 15, 6, *which come from row 6 of Pascal's Triangle.* $\left(a+b\right)^n = \left(5+3\right)^6$, *and the term immediately preceding* 67500 *is* $15 \cdot \left(3^2 \cdot 5^4\right) = 15 \cdot 9 \cdot 625 = \boxed{84375}$.

2. *Inscribe* $\triangle XYZ$ *in circle C (Figure 2). Let* $\overline{XZ}$ *belong to the 21-gon,* $\overline{XY}$ *to the 28-gon, and* $\overline{YZ}$ *to the n-gon. (This will allow* $\overline{YZ}$ *to be the smallest side of the triangle, and force n to be as large as possible.) Focusing on the central angles at C:*

$$m\angle XCY + m\angle YCZ = m\angle XCZ \Rightarrow \frac{360°}{28} + \frac{360°}{n} = \frac{360°}{21} \Rightarrow \frac{1}{4 \cdot 7} + \frac{1}{n} = \frac{1}{3 \cdot 7} \Rightarrow 3 + \frac{3 \cdot 4 \cdot 7}{n} = 4 \Rightarrow \frac{84}{n} = 1 \Rightarrow n = \boxed{84}.$$

3. *The center of each circle will have coordinates* (*r*, *r*), *where r is the length of the circle's radius. Using the formula for the distance from a point to a line,* $r = \frac{\left|3r + 2r - 6\right|}{\sqrt{3^2 + 2^2}} = \frac{\left|5r - 6\right|}{\sqrt{13}} \Rightarrow \left|5r - 6\right| = r\sqrt{13} \Rightarrow 5r - 6 = \pm r\sqrt{13} \Rightarrow r = \frac{6}{5 \pm \sqrt{13}}$.

The difference in r-values is $\frac{6}{5-\sqrt{13}} - \frac{6}{5+\sqrt{13}} = \frac{6\left(5+\sqrt{13}\right) - 6\left(5-\sqrt{13}\right)}{\left(5-\sqrt{13}\right)\left(5+\sqrt{13}\right)} = \frac{12\sqrt{13}}{25-13} = \boxed{\sqrt{13}}$.

Will the difference <u>always</u> be $\sqrt{a^2+b^2}$ ***? Is there a theorem here? What about the <u>sum</u> of the lengths of the radii?***

4. *Draw radius* $\overline{ST}$, *with T being the semicircle's point of tangency on* $\overline{AB}$. *Since* $\overline{BT}$ *and* $\overline{BC}$ *are both tangents to the same circle from the same point, they must be equal in length. Thus AT = 1. If r = length of semicircle's radius, then AS = 3 – r. Now, in right* $\triangle AST$, *we have* $1^2 + r^2 = \left(3-r\right)^2 \Rightarrow 1 + \cancel{r^2} = 9 - 6r + \cancel{r^2} \Rightarrow 6r = 8 \Rightarrow r = \boxed{\frac{4}{3}}$.

5. *Let f = time it takes to fill the brush, p = time to paint 1 foot, and w = time to walk 1 foot. As the painter works, his progress is given by* $\left(f+p+w\right) + \left(f+p+3w\right) + \left(f+p+5w\right) + \ldots$ *After 1 hour, the total time will include the terms 100 f & 100p, so the series w + 3w + 5w + ... will have 100 terms, and sum to the 100th square number* $\left(100^2\right)$. *Similarly, during Hour #2, the total time will be* $50f + 50p + \left(201 + 203 + \ldots + 299\right)w = 50f + 50p + 12500w$. *Setting Hour #1 and Hour #2's expressions equal,* $50f + 50p = 2500w \Rightarrow 100f + 100p + 100^2 w = 2\left(2500w\right) + 10000w = 15000w$, *so the painter walks 15,000 feet in an hour. During Hour 3,* $50f + 50p + \left(301 + 303 + \ldots + 399\right)w = 50f + 50p + 17500w = 20000w$; *i.e., the painter needs additional time equivalent to him walking 20,000 feet, which is* $\boxed{\text{1 hour, 20 minutes}}$.

6. *Let the intercepts of f(x) be (a, 0) and (0, b). Then the intercepts of g(x) are (–a, 0) and (0, –b). The slopes of f and g are equal, so the lines are parallel. Let f(x) = mx + b, g(x) = mx – b. Then* $f\left(g\left(x\right)\right) = m\left(mx - b\right) + b = m^2x - mb + b = 9x + 12$, *so* $m^2 = 9 \Rightarrow m = \pm 3$, *and* $b\left(1-m\right) = 12 \Rightarrow (m,b) = (-3,3)$ *or* $(m,b) = (3,-6)$. *Reversing the assignments of the f and g equations results in the same two ordered pairs. So* $f\left(10\right) = -3\left(10\right) + 3 = \boxed{-27}$, *or* $f\left(10\right) = 3\left(10\right) - 6 = \boxed{24}$.