Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

________________ 1. The trip from Albert Lea to Bemidji requires 4.5 hours when traveling at an average speed of 70 mph. How many hours does the same trip require when traveling at an average speed of 60 mph?

k = ____________ 2. If the graph of $ax + by = 8$ passes through $\left(3, \frac{1}{b}\right)$ and $\left(-\frac{1}{a}, -5\right)$, then it will also pass through the point (15, *k*). Determine exactly the value of *k*.

# of girls = ________ 3. When the 15-member girls basketball team leaves from a party, there remain at the party 2 boys for every girl. After this, the 45-member boys football team leaves, and there remain 5 girls for every boy. How many girls were at the party in the beginning?

product = ________ 4. Two distinct positive integers are chosen such that their difference, sum, and product (in that order) are in the ratio $1 : 7 : 36$. Calculate the product of the two numbers.

Name: ____________________________ Team: ____________________________

Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event A

SOLUTIONS

5.25

1. The trip from Albert Lea to Bemidji requires 4.5 hours when traveling at an average speed of 70 mph. How many hours does the same trip require when traveling at an average speed of 60 mph?

$d = rt = (70)(4.5) = 315$ miles, *so* $d = rt \Rightarrow 315 = 60t \Rightarrow t = 5.25$ hours.

$k =$ **15**

2. If the graph of $ax + by = 8$ passes through $\left(3, \frac{1}{b}\right)$ and $\left(-\frac{1}{a}, -5\right)$, then it will also pass through the point (15, k). Determine exactly the value of k.

Substituting the coordinates of the first point: $ax + by = 8 \Rightarrow 3a + 1 = 8 \Rightarrow a = \frac{7}{3}$.

Substituting the coordinates of the second point: $ax + by = 8 \Rightarrow -1 - 5b = 8 \Rightarrow b = -\frac{9}{5}$.

Substituting (15, k) into $\frac{7}{3}x - \frac{9}{5}y = 8$ *yields* $35 - \frac{9}{5}k = 8 \Rightarrow -\frac{9}{5}k = -27 \Rightarrow k = 15$.

# of girls = **40**

3. When the 15-member girls basketball team leaves from a party, there remain at the party 2 boys for every girl. After this, the 45-member boys football team leaves, and there remain 5 girls for every boy. How many girls were at the party in the beginning?
[adapted from MAA: <u>The Contest Problem Book I</u>]

The problem statement leads to the system $\begin{cases} b = 2(g-15) \\ g - 15 = 5(b-45) \end{cases}$. *Substituting using (g − 15),*

$b = 2[5(b-45)] \Rightarrow b = 10b - 450 \Rightarrow b = 50$. *By the first equation,* $50 = 2(g-15) \Rightarrow g = 40$.

product = **108**

4. Two distinct positive integers are chosen such that their difference, sum, and product (in that order) are in the ratio 1 : 7 : 36. Calculate the product of the two numbers.
[adapted from MAA: <u>The Contest Problem Book II</u>]

Let (m, n) represent the two numbers, with m > n, and let d = their difference. We can then use the given ratio to create the system $\begin{cases} m - n = d \\ m + n = 7d \end{cases}$ *, which can be solved quickly by elimination to discover that (m, n) = (4d, 3d). The product mn is therefore* $12d^2$*, but is also 36d by the given ratio. Setting* $12d^2 = 36d$ *yields d = 3 as the only viable solution, so the product is 36(3) = 108.*

Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

________________ 1. The midpoints of the sides of a square are joined to form an inscribed square. If the sides of the original square have length 1, determine exactly the area of the inscribed square.

$AC =$ ________________ 2. Parallelogram *ABCD (Figure 2)* has sides of length $AD = 4$ and $AB = 7$, and short diagonal $BD = 5$. Determine exactly the length of $\overline{AC}$.

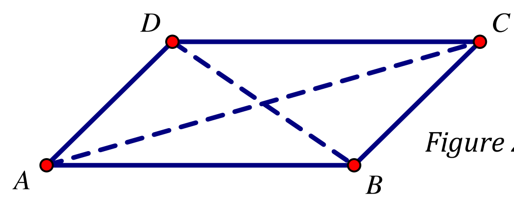


Figure 2

________________ 3. Square *SQUA (Figure 3)* has sides of length 1, with $QZ = QY = \frac{3}{4}$ and $AW = AX = \frac{1}{3}$. Calculate the shortest distance between segments $\overline{WX}$ and $\overline{YZ}$.

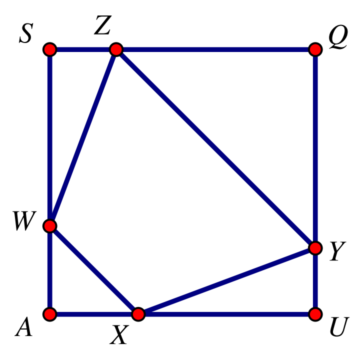


Figure 3

________________ 4. Referring again to *Figure 2*, calculate the length of the altitude from D to $\overline{AB}$.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

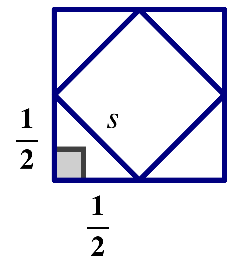
2010-11 Meet 3, Individual Event B

SOLUTIONS

Answer: $\boxed{\frac{1}{2}}$

1. The midpoints of the sides of a square are joined to form an inscribed square. If the sides of the original square have length 1, determine exactly the area of the inscribed square.

Area of inscribed square = $s^2 = \left(\frac{1}{2}\cdot\sqrt{2}\right)^2 = \frac{1}{4}\cdot 2 = \frac{1}{2}$.



Answer: $AC = \boxed{\sqrt{105}}$

2. Parallelogram *ABCD (Figure 2)* has sides of length *AD* = 4 and $\overline{AB}$ = 7, and short diagonal *BD* = 5. **Determine exactly** the length of $\overline{AC}$.

A theorem of parallelograms states that the sum of the squares of the four sides equals the sum of the squares of the diagonals. In this case:
$2\left(4^2\right)+2\left(7^2\right)=5^2+AC^2 \;\Rightarrow\; AC=\sqrt{105}$.

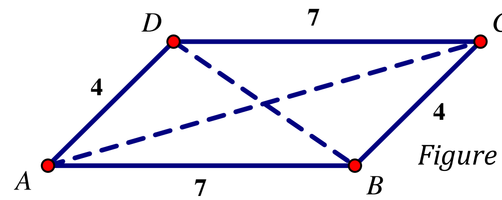


Figure 2

Answer: $\boxed{\frac{11\sqrt{2}}{24}}$ **or** $\boxed{\approx 0.648}$

3. Square *SQUA (Figure 3)* has sides of length 1, with $QZ = QY = \frac{3}{4}$ and $AW = AX = \frac{1}{3}$. Calculate the shortest distance between segments $\overline{WX}$ and $\overline{YZ}$.

Since $\overline{WX}$ *and* $\overline{YZ}$ *are both at 45° angles to the horizontal, they are parallel. Thus WXYZ is a trapezoid, and we seek its height.*

Noting $AQ=\sqrt{2}$, $AM=\frac{1}{3}\div\sqrt{2}=\frac{1}{6}\sqrt{2}$, *and* $NQ=\frac{3}{4}\div\sqrt{2}=\frac{3}{8}\sqrt{2}$,

$MN = AQ - AM - NQ = \sqrt{2}-\frac{1}{6}\sqrt{2}-\frac{3}{8}\sqrt{2}=\frac{11}{24}\sqrt{2}$.

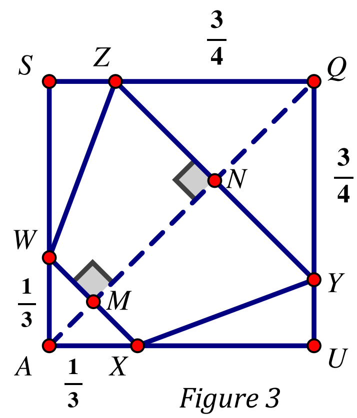


Figure 3

Answer: $\boxed{\frac{8\sqrt{6}}{7}}$ **or** $\boxed{\approx 2.799}$

4. Referring again to *Figure 2*, calculate the length of the altitude from *D* to $\overline{AB}$.

By Heron's Formula, the semiperimeter of $\triangle ABD$ *is* $\frac{4+5+7}{2}=\frac{16}{2}=8$, *and therefore*

$Area[\triangle ABD]=\sqrt{8(8-4)(8-5)(8-7)}=\sqrt{96}=4\sqrt{6}$. *Also,* $Area[\triangle ABD]=\frac{1}{2}bh=\frac{7}{2}h$.

Setting these two expressions equal, $4\sqrt{6}=\frac{7}{2}h \;\Rightarrow\; h=\frac{8\sqrt{6}}{7}$.

Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

______________ 1. How many solutions to $\sin\theta = -\cos\theta$ exist on the interval $0 \le \theta < 2\pi$?

$BC =$ ______________ 2. In isosceles $\triangle ABC$ *(Figure 2)*, $AB = AC = 4$ and $m\angle A = 25°$. Calculate the length BC.

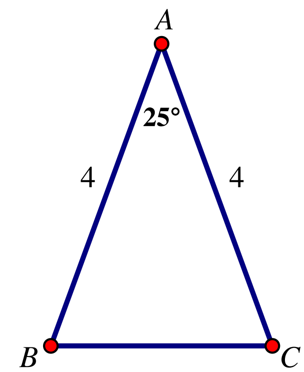


Figure 2

$m\angle N =$ ______________ 3. Pentagon *QUINT (Figure 3)* is equilateral, with a right angle at Q. If $\overline{TU} \parallel \overline{NI}$, calculate the measure of $\angle N$, in degrees.

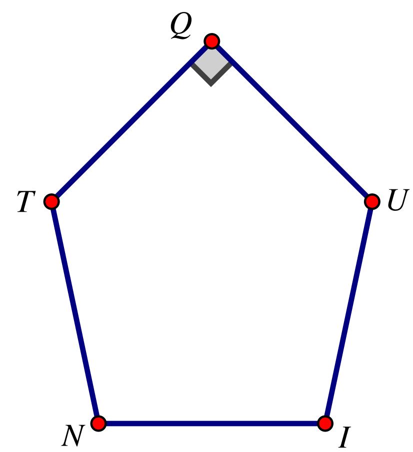


Figure 3

$x =$ ______________ 4. Determine exactly the x-coordinate of the point of intersection of the graphs of

$$f(x) = \cos^{-1} x \text{ and } g(x) = \tan^{-1} x.$$

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event C

SOLUTIONS

2

1. How many solutions to $\sin\theta = -\cos\theta$ exist on the interval $0 \le \theta < 2\pi$?

$\sin\theta = -\cos\theta \;\Rightarrow\; \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\cos\theta}{\cos\theta} \;\Rightarrow\; \tan\theta = -1$, *which occurs twice on the unit circle.*

$BC \approx$ **1.732** *or* **1.731**

2. In isosceles $\triangle ABC$ *(Figure 2)*, $AB = AC = 4$ and $m\angle A = 25°$. Calculate the length BC.

$$\frac{\sin 77.5°}{4} = \frac{\sin 25°}{BC} \;\Rightarrow\; BC = \frac{4\sin 25°}{\sin 77.5°} \approx 1.732.$$

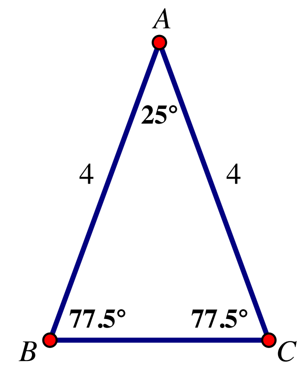


Figure 2

$m\angle N \approx$ **101.953°** *or* **101.952°**

3. Pentagon *QUINT (Figure 3)* is equilateral, with a right angle at Q. If $\overline{TU} \parallel \overline{NI}$, calculate the measure of $\angle N$, in degrees.

Label the figure as shown, with $m\angle N = \alpha$. *Applying the Law of Cosines in two different ways to* $\overline{IT}$ *:*

$$IT^2 = x^2 + x^2 - 2x^2\cos\alpha = x^2 + 2x^2 - 2x^2\sqrt{2}\cos\left(180° - \alpha\right)$$

$$2x^2\left(1 - \cos\alpha\right) = x^2\left(3 - 2\sqrt{2}\cdot\left(-\cos\alpha\right)\right)$$

$$2 - 2\cos\alpha = 3 + 2\sqrt{2}\cos\alpha$$

$$-1 = \left(2 + 2\sqrt{2}\right)\cos\alpha$$

So $\cos\alpha = \dfrac{-1}{2\sqrt{2}+2}$, *and* $\alpha \approx 101.953°$.

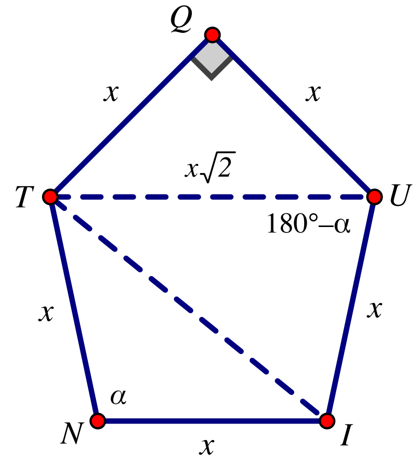


Figure 3

$x =$ $\sqrt{\dfrac{\sqrt{5}-1}{2}}$ *or* $\dfrac{\sqrt{2\sqrt{5}-2}}{2}$

4. **Determine exactly** the x-coordinate of the point of intersection of the graphs of $f\left(x\right) = \cos^{-1} x$ and $g\left(x\right) = \tan^{-1} x$.

$$\cos^{-1} x = \tan^{-1} x = y \;\Rightarrow\; x = \cos y = \tan y = \frac{\sin y}{\cos y} \;\Rightarrow\; \cos^2 y = \sin y \;\Rightarrow\; 0 = \sin^2 y + \sin y - 1$$

$\Rightarrow\; \sin y = \dfrac{-1 \pm \sqrt{1 - 4\left(1\right)\left(-1\right)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$. *Only the "+" solution is valid, so* $\sin y = \dfrac{-1+\sqrt{5}}{2}$.

$$x = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{2}\right)^2} = \sqrt{\frac{4}{4} - \left(\frac{6-2\sqrt{5}}{4}\right)} = \sqrt{\frac{2\sqrt{5}-2}{4}} = \sqrt{\frac{\sqrt{5}-1}{2}}.$$

Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. Determine exactly the value of $\left(-\frac{1}{2}\right)^4 + \left(4\right)^{-\frac{1}{2}}$.

________________ 2. Determine exactly the value of $\frac{\log_4 32}{\log_8 16}$.

________________ 3. Let $f\left(x\right) = \log_2\left(7 - \log_2 x\right)$. Determine the number of integers in the domain of $f(x)$.

$n =$ ________________ 4. Determine exactly all real values of n that satisfy $\sqrt{3 + \log_8 n} = \sqrt[4]{9 + \log_2 n}$.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2010-11 Meet 3, Individual Event D

SOLUTIONS

$\boxed{\frac{9}{16}}$ *or* $\boxed{0.5625}$

1. **Determine exactly** the value of $\left(-\frac{1}{2}\right)^4+\left(4\right)^{-\frac{1}{2}}$.

$$\left(-\frac{1}{2}\right)^4+\left(4\right)^{-\frac{1}{2}}=\frac{1}{16}+\frac{1}{\sqrt{4}}=\frac{1}{16}+\frac{1}{2}=\frac{9}{16}.$$

$\boxed{\frac{15}{8}}$ *or* $\boxed{1.875}$

2. Determine exactly the value of $\dfrac{\log_4 32}{\log_8 16}$.

Applying the change-of-base property, $\dfrac{\log_4 32}{\log_8 16}=\dfrac{\left(\dfrac{\log_2 32}{\log_2 4}\right)}{\left(\dfrac{\log_2 16}{\log_2 8}\right)}=\dfrac{\left(\dfrac{5}{2}\right)}{\left(\dfrac{4}{3}\right)}=\dfrac{5}{2}\cdot\dfrac{3}{4}=\dfrac{15}{8}$

$\boxed{127}$

3. Let $f\left(x\right)=\log_2\left(7-\log_2 x\right)$. Determine the number of integers in the domain of $f(x)$.

For the inner logarithm to exist, we need $x>0$. *For the outer logarithm to exist, we need* $7-\log_2 x>0 \Rightarrow \log_2 x<7 \Rightarrow x<2^7=128$. *So* $0<x<128$, *an interval that contains 127 integers.*

$n=\boxed{\frac{1}{512},1}$

4. Determine exactly all real values of n that satisfy $\sqrt{3+\log_8 n}=\sqrt[4]{9+\log_2 n}$.

Noting first that $\log_8 n=\dfrac{\log_2 n}{\log_2 8}=\dfrac{\log_2 n}{3} \Rightarrow \log_2 n=3\cdot\log_8 n$, *let* $a=\log_8 n$.

Then $\sqrt{3+\log_8 n}=\sqrt[4]{9+\log_2 n} \Rightarrow \sqrt{3+a}=\sqrt[4]{9+3a} \Rightarrow \sqrt[4]{\left(3+a\right)^2}=\sqrt[4]{9+3a}$.

Raising both sides to the 4th power, $9+6a+a^2=9+3a \Rightarrow a\left(a+3\right)=0 \Rightarrow a=-3,0$.

So $\log_8 n=-3 \Rightarrow n=8^{-3}=\dfrac{1}{512}$, *and* $\log_8 n=0 \Rightarrow n=8^0=1$.

Minnesota State High School Mathematics League

2010-11 Meet 3, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

____________ 1. Find all ordered pairs of integers (x, y) for which $(x + 2y)(2x + y) = 27$.

____________ 2. Rectangle *RECT (Figure 2)* has sides of length 12 and 16. Points Z and O are midpoints of two of the sides, while points I and D subdivide the sides on which they are placed in the ratio 1 : 3 (with $RD > DT$, and $IC > TI$). Determine exactly the height of trapezoid *ZOID*.

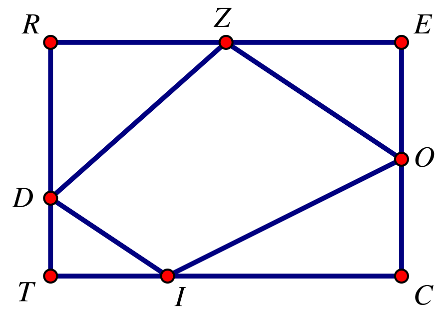


Figure 2

____________ 3. Find all pairs (α, β) of acute angles that satisfy $\sin(2\alpha + \beta) = \cos(\alpha - 2\beta) = \sin(\alpha + 50°)$. *(Express all angle measures in degrees.)*

$x =$ ____________ 4. Determine exactly the value of x for which $\log_2 4x + \log_4 8x + \log_8 2x = 4$.

____________ 5. Determine the number of pairs (c, d) of integers, with $1 < c \le d \le 100$, for which $\log_c d$ is a rational number.

____________ 6. A geometry book gives the following instructions for a construction:

" A circle has center O, radius r, and perpendicular diameters $\overline{AB}$ and $\overline{CD}$.

- Construct M as the midpoint of $\overline{OB}$.
- Construct an arc, centered at M with radius $\overline{MC}$, cutting $\overline{OA}$ at E.
- Draw chords $\overline{CP}$ and $\overline{CS}$, each having length CE. "

Note that $\triangle CPS$ is isosceles. Determine exactly, in terms of r, the equal lengths CP and CS.

Team: ____________________

Minnesota State High School Mathematics League

2010-11 Meet 3, Team Event

SOLUTIONS (page 1)

Answer: $(-5,1)$ $(-1,5)$ $(1,-5)$ $(5,-1)$

1. Find all ordered pairs of integers (x, y) for which $(x + 2y)(2x + y) = 27$.
[The 1st High School Math League Problem Book, Conrad & Flegler]

(Award 1 point for each correct ordered pair)

Answer: 12

2. Rectangle *RECT (Figure 2)* has sides of length 12 and 16. Points *Z* and *O* are midpoints of two of the sides, while points *I* and *D* subdivide the sides on which they are placed in the ratio 1 : 3 (with $RD > DT$, and $IC > TI$).
Determine exactly the height of trapezoid *ZOID*.

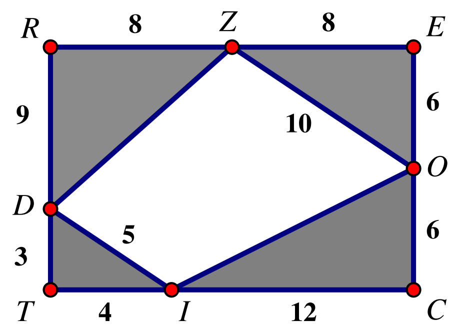


Figure 2

Answer: $\left(\frac{75}{2}, \frac{35}{2}\right)$ $(30, 20)$ $\left(\frac{110}{3}, 20\right)$ $(35, 15)$

3. Find all pairs (α, β) of acute angles that satisfy $\sin(2\alpha + \beta) = \cos(\alpha - 2\beta) = \sin(\alpha + 50°)$.
(Express all angle measures in degrees.)

(Award 1 point for each correct ordered pair)

Answer: $x = \sqrt[11]{2}$

4. **Determine exactly** the value of *x* for which $\log_2 4x + \log_4 8x + \log_8 2x = 4$.

Answer: 124

5. Determine the number of pairs (c, d) of integers, with $1 < c \le d \le 100$, for which $\log_c d$ is a rational number.

Answer: $\frac{\sqrt{10-2\sqrt{5}}}{2}r$

6. A geometry book gives the following instructions for a construction:
" A circle has center *O*, radius *r*, and perpendicular diameters $\overline{AB}$ and $\overline{CD}$.
- Construct *M* as the midpoint of $\overline{OB}$.
- Construct an arc, centered at *M* with radius $\overline{MC}$, cutting $\overline{OA}$ at *E*.
- Draw chords $\overline{CP}$ and $\overline{CS}$, each having length *CE*. "

Note that $\triangle CPS$ is isosceles. Determine exactly, in terms of *r*, the equal lengths *CP* and *CS*.

Minnesota State High School Mathematics League

2010-11 Meet 3, Team Event

SOLUTIONS (page 2)

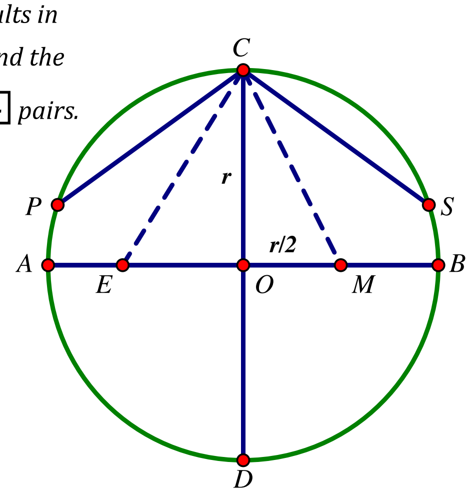
1. *Let (a, b) represent a pair of factors whose product is 27. WLOG, form the system of equations* $\begin{cases} 2x+y=a \\ x+2y=b \end{cases}$. *Adding the two equations yields 3x + 3y = a + b, and since x and y are integers, the sum of the factors must be divisible by 3. This eliminates any combination of* ± 1 *and* ± 27, *leaving only (a, b) =* (−3, −9), (3, 9), (−9, −3), and (9, 3). *Forming and solving a system for each of those pairs of factors results in four solutions:* $\boxed{\{(-5,1),(-1,5),(1,-5),(5,-1)\}}$.

2. *Label Figure 2 as shown. Area[ZOID] = Area[RECT] − Area[RDZ] − Area[ZEO] − Area[OCI] − Area[TID]*
$$=(12)(16)-\frac{1}{2}(9)(8)-\frac{1}{2}(8)(6)-\frac{1}{2}(6)(12)-\frac{1}{2}(3)(4)=192-36-24-36-6,$$
So $\frac{5+10}{2}\cdot h=192-36-24-36-6 \Rightarrow \frac{15}{2}h=90$, *and the height of ZOID is* $\boxed{12}$.

3. *Since* $\sin(\alpha+50°)$ *is positive for all acute angles* α, *the other two terms of the equality must also be positive; namely,* $2\alpha+\beta<180°$ *and* $\alpha-2\beta<90°$. *Treating the left-most equality as cofunctions, either* $(2\alpha+\beta)+(\alpha-2\beta)=90°$ *or* $(2\alpha+\beta)-(\alpha-2\beta)=90°$. *The "outside" equality requires that either* $2\alpha+\beta=\alpha+50°$ *or* $(2\alpha+\beta)+(\alpha+50°)=180°$.
Rewrite these 4 equations into a system: $\begin{cases} 3\alpha-\beta=90° \text{ or } \alpha+3\beta=90° \\ \alpha+\beta=50° \text{ or } 3\alpha+\beta=130° \end{cases}$
Solving, using every possible pairing of equations, yields the pairs $\boxed{\left\{(35,15),(30,20),\left(\frac{110}{3},20\right),\left(\frac{75}{2},\frac{35}{2}\right)\right\}}$.

4. $\log_2 4x+\log_4 8x+\log_8 2x=4 \Rightarrow \log_2 4+\log_2 x+\frac{\log_2 8+\log_2 x}{\log_2 4}+\frac{\log_2 2+\log_2 x}{\log_2 8} \Rightarrow 2+\log_2 x+\frac{3+\log_2 x}{2}+\frac{1+\log_2 x}{3}$
Setting $y=\log_2 x$, *we have* $2+y+\frac{3}{2}+\frac{y}{2}+\frac{1}{3}+\frac{y}{3}=4 \Rightarrow \frac{23}{6}+\frac{11}{6}y=4 \Rightarrow 23+11y=24 \Rightarrow y=\frac{1}{11}$, *and x =* $\boxed{2^{\frac{1}{11}}}$.

5. *Each of the 99 pairs* (2, 2), (3, 3), . . . , (100, 100) *result in a rational number. Additionally, choosing any two distinct powers of 2 from* {2, 4, 8, 16, 32, 64} *results in* ${}_6C_2=15$ *additional valid pairs. Similarly for the powers of 3* (${}_4C_2=6$ *pairs) and the powers of 5, 6, 7, and 10* (*2 pairs each). Total is 99 + 15 + 6 + 2 + 2 + 2 + 2 =* $\boxed{124}$ *pairs.*



6. *By the Pythagorean Theorem,* $CM=ME=\sqrt{r^2+\left(\frac{r}{2}\right)^2}=\sqrt{\frac{5r^2}{4}}=\frac{\sqrt{5}}{2}r$.
So $OE=ME-OM=\frac{\sqrt{5}}{2}r-\frac{r}{2}=\frac{\sqrt{5}-1}{2}r$, *and* $CE=CP=CS=\sqrt{OE^2+OC^2}$
$$=\sqrt{r^2+\left(\frac{\sqrt{5}-1}{2}r\right)^2}=\sqrt{r^2+\left(\frac{5-2\sqrt{5}+1}{4}\right)r^2}=\sqrt{\frac{10-2\sqrt{5}}{4}r^2}=\boxed{\frac{\sqrt{10-2\sqrt{5}}}{2}r}.$$