Minnesota State High School Mathematics League

Individual Event

2009-10 Event 5A
SOLUTIONS

1. Compute, in simplified form, the value of the 8th expression in this pattern:

   1   2 + 3   4 + 5 + 6   7 + 8 + 9 + 10   ...

$\boxed{260}$.

The next expression in the pattern sums 11 through 15, the 6th expression sums 16 through 21, the 7th expression sums 22 through 28, and so the 8th expression = 29 + ... + 36 = 260.

2. In the pattern ..., *a*, 2, *b*, *a*, ..., each number is equal to twice the previous number plus the number two places back. Compute the value of *a*.

***a* =** $\boxed{-10}$.

By definition, b = 2(2) + a = 4 + a, and a = 2(b) + 2. Substituting for the value of b, $a = 2(4+a)+2 = 10+2a$*, and a = –10.*

3. The product of two numbers is equal to their sum. If we add 1 to both numbers, their new product will be five times their original product. What is the original sum?

$\boxed{\frac{1}{3}}$.

Let the two numbers be x and y, and let their sum be S = x + y = xy. The second statement tells us that (x + 1)(y + 1) = 5xy. Expanding the left side: xy + x + y + 1 = 5xy.
Now replace every instance of xy or (x + y) with S: $S+S+1=5S \Rightarrow S=\frac{1}{3}$.

4. The product $\underline{3}\,\underline{4}\,\underline{5}\cdot\underline{5}\,\underline{6}\,\underline{7}$ is not a perfect square, but changing just one of the six underlined digits creates a product of N^2, where *N* is a positive integer. Compute *N*.

***N* =** $\boxed{441}$ **.**

$345 = 3\cdot 5\cdot 23$*, and* $567 = 7\cdot 9^2$*. If we leave the first factor unchanged, then the second factor would need to be of the form* $(3\cdot 5\cdot 23)\cdot k^2$ *for some positive integer k, which is impossible. So we leave the second factor unchanged, meaning the first factor must be of the form* $7\cdot k^2$*. Furthermore, the digit to be changed must be the 5, since the only three-digit number of the proper form which ends in 5 is* $7\cdot 5^2 = 175$*, which would require two changed digits. The only integer in the 340s divisible by 7 is* $343 = 7\cdot 7^2$*, resulting in a product of* $N^2 = (7\cdot 7^2)(7\cdot 9^2) = 7^4\cdot 9^2$*. So* $N = 7^2\cdot 9 = 441$.

Minnesota State High School Mathematics League

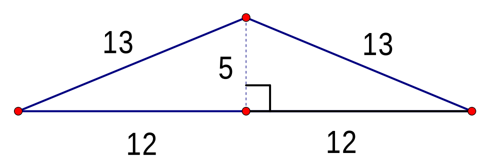
Individual Event

2009-10 Event 5B SOLUTIONS

1. Calculate the area of the isosceles triangle with side lengths 13, 13, and 24.

$\boxed{60}$.

Draw the triangle, dropping the altitude as shown. This creates two 5-12-13 right triangles, so the area is $\frac{1}{2}(24)(5)=60$.



2. If the numerical value of the area of a square plus two times the numerical value of its perimeter is equal to 20, what is the length of one side of the square?

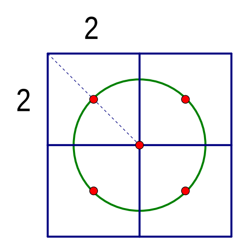
$\boxed{2}$.

Let s be the desired side length. $s^2+2(4s)=20 \Rightarrow s^2+8s-20=(s+10)(s-2)=0$.
Eliminating s = –10 leaves s = 2. ***(Mathematics Teacher, September 2006)***

3. A square with an area of 16 is subdivided into four congruent smaller squares. Calculate the area of the circle that passes through the centers of all four smaller squares.

$\boxed{2\pi}$ **, or** $\boxed{\approx 6.283}$.

The large square has a side length of 4, so the smaller squares have side length 2. Each diagonal of the smaller squares has length $2\sqrt{2}$ *, so the radius of the circle is* $\sqrt{2}$ *, and its area is* $\pi\left(\sqrt{2}\right)^2=2\pi$. ***(Mathematics Teacher, April 2006)***



4. The base of a right square pyramid is inscribed in the base of a cylinder. The height of the pyramid equals the side length of its base, and is also one-third the height of the cylinder. Find the ratio of the surface area of the pyramid to the surface area of the cylinder.

$\boxed{\dfrac{1+\sqrt{5}}{\pi\left(1+3\sqrt{2}\right)}}$ **,**

or $\boxed{\approx 0.196 : 1}$ **,**

or $\boxed{\approx 1 : 5.090}$.

Let h = height of pyramid. Then 3h = height of cylinder. If r is the radius of the cylinder, then the side length of the square base is $r\sqrt{2}$ *, which equals h.*

$SA_{cylinder}=2\pi r(r+h)=2\pi r\left(r+3r\sqrt{2}\right)=2\pi r^2\left(1+3\sqrt{2}\right)$;

$$SA_{pyramid}=\tfrac{1}{2}Pl+B=\tfrac{1}{2}\left(4r\sqrt{2}\right)\left(\sqrt{\left(\tfrac{r\sqrt{2}}{2}\right)^2+h^2}\right)+\left(r\sqrt{2}\right)^2=2r\sqrt{2}\left(\sqrt{\tfrac{r^2}{2}+2r^2}\right)+2r^2$$

$$=2r\sqrt{r^2+4r^2}+2r^2=2r^2\sqrt{5}+2r^2=2r^2\left(\sqrt{5}+1\right).$$

So $SA_{pyramid}:SA_{cylinder}=\cancel{2r^2}\left(\sqrt{5}+1\right):\cancel{2}\pi\cancel{r^2}\left(1+3\sqrt{2}\right)=1+\sqrt{5}:\pi\left(1+3\sqrt{2}\right)$.

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 5C
SOLUTIONS

1. Three standard 6-sided dice are rolled. Express, as the quotient of two relatively prime integers, the probability that the numbers showing on the top faces of the dice sum to 4.

$\boxed{\frac{1}{72}}$.

One die must show a 2, and there are three ways this can happen (1-1-2; 1-2-1; 2-1-1); the other two dice are showing 1's. The probability is $\frac{3}{6\cdot 6\cdot 6}=\frac{1}{72}$.

2. There are 14 applicants for a position. A screening committee is to present to the manager three candidates, listed in alphabetical order to avoid bias. How many such lists are possible?

$\boxed{364}$.

At first glance, this may seem like a permutation problem...but there is only <u>one</u> acceptable ordering that can result from each subset of three names. Therefore, we are looking for $\binom{14}{3}=\frac{14\cdot 13\cdot \overset{2}{\cancel{12}}}{\cancel{3\cdot 2\cdot 1}}=364$.

3. How many unique license plates can be made using two letters (excluding the letter "O", which looks too much like a zero) and two single-digit numbers? Neither letters nor numbers need to be distinct, and note that plates like A1B0 and 0B1A are both unique.

$\boxed{375000}$.

There are four positions, and we need to choose two of them in which to place letters: $\binom{4}{2}=6$. *(The numbers go in the two unchosen positions.) Then, we have 25 options for each letter, and 10 options for each single-digit #.* $6\cdot(25\cdot 25)(10\cdot 10)=375000$.

4. Evaluate $\frac{2^{-n}(2n+1)!}{n!\,[1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)]}$ for n = 10,000.

$\boxed{20001}$.

First, move the 2^{-n} *to the denominator, using the n factors of 2 to double each factor of n! . This results in:* $\frac{(2n+1)!}{[1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)]\,[2\cdot 4\cdot 6\cdot\ldots\cdot(2n)]}$. *The denominator now contains all of the factors necessary to create (2n)! , so we have* $\frac{(2n+1)!}{(2n)!}=2n+1$.

Evaluating for n = 10,000: 2n + 1 = 2(10,000) + 1 = 20,000 + 1 = 20,001.

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 5D
SOLUTIONS

1. What number is one-fourth of the way from $\frac{2}{5}$ to $\frac{3}{5}$?

$\boxed{\frac{9}{20}}$***, or*** $\boxed{0.45}$.

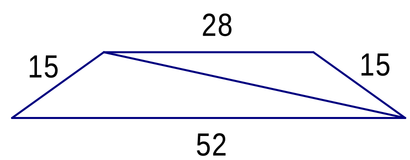
$$\boxed{\frac{2}{5}+\left(\frac{3}{5}-\frac{2}{5}\right)\cdot\frac{1}{4}=\frac{2}{5}+\frac{1}{20}=\frac{8}{20}+\frac{1}{20}=\frac{9}{20}.}$$

(from AMC 12A, #3)

2. An isosceles trapezoid has legs of length 15 and bases of lengths 28 and 52. Find the length of one of the diagonals, given that this length is a prime number.

$\boxed{41}$.

The diagonal (call it d) divides the isosceles trapezoid into two triangles. Applying the Triangle Inequality to each triangle: $52-15<d<28+15 \;\Rightarrow\; 37<d<43$.
Since d is prime, d must be 41.



(from AMC 12A, #10)

3. The first three terms of an infinite geometric sequence are x^2+3x+2, x^2+5x+6, and $x^2+8x+15$, respectively. Compute the value of the term in the sequence which is *closest*, numerically, to 2010.

$\boxed{1536}$.

(from AMC 12A, #7)

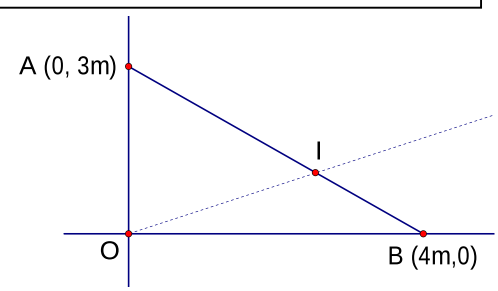
Factor the first three terms, rewriting them as $(x+1)(x+2)$, $(x+2)(x+3)$, *and* $(x+3)(x+5)$. *Since there is a constant ratio between successive terms,*
$\frac{(x+1)\cancel{(x+2)}}{\cancel{(x+2)}(x+3)}=\frac{(x+2)\cancel{(x+3)}}{\cancel{(x+3)}(x+5)} \;\Rightarrow\; (x+3)(x+2)=(x+1)(x+5) \;\Rightarrow\; x^2+5x+6=x^2+6x+5$,
and x = 1. So the sequence proceeds: 6, 12, 24, 48, 96, 192, 384, 768, <u>1536</u>, 3072...

4. A triangle has vertices (0, 0), (0, 3*m*), and (4*m*, 0), and the line $y = mx$ divides the triangle into two triangles of equal perimeter. Find the value of *m*.

m **=** $\boxed{\frac{1}{2}}$.

(from AMC 12A, #14)

AB = 5m. Let point I be p percent of the way from A to B. With $\overline{OI}$ shared, AO+AI = IB+OB
$\Rightarrow\; 3m+p(5m)=4m+(1-p)(5m) \;\Rightarrow\; p=\frac{3}{5}$.
So $I=\left(p\cdot OB,(1-p)\cdot OA\right)=\left(\frac{12}{5}m,\,\frac{6}{5}m\right)$, and
$m=\frac{y}{x}=\frac{6/5}{12/5}=\boxed{\frac{1}{2}}$.



Notice that we form two triangles of equal <u>area</u> by creating isosceles $\triangle AOI$ with AI=IO, but we form two triangles of equal <u>perimeter</u> by creating isosceles $\triangle AOI$ with AI=AO. Is there a theorem here, similar to the theorem that involves area equality and medians, that extends perimeter equality to <u>non-right</u> triangles?

Minnesota State High School Mathematics League

Team Event

2009-10 Meet 5
SOLUTIONS

1. Consider a cube of side length 1. The centers of each pair of faces of the cube that share a common edge are connected to form a regular octahedron (a regular polyhedron comprised of 8 congruent equilateral triangular faces). Compute the volume of this octahedron.

$\boxed{\frac{1}{6}}$. ***(Mathematics Teacher, Feb. 2006)***

2. The sides of a triangle have lengths 6, 7, and x. What is the largest possible area of such a triangle? ***(Mathematics Teacher, May 2006)***

$\boxed{21}$.

3. Claire designed a 3 x 3 magic square, where every row, column, and diagonal sums to the same total. Then, to every number in the square, she randomly added or subtracted a number d . The resulting grid is shown in *Figure 3*. Compute d.

109	20	33
18	70	122
59	96	7

Figure 3

d = $\boxed{12}$.

4. In Event A, students were asked to find the value of the 8th expression in the pattern:

1   2 + 3   4 + 5 + 6   7 + 8 + 9 + 10   ...

Find the value of the 100th expression in the pattern.

$\boxed{500050}$.

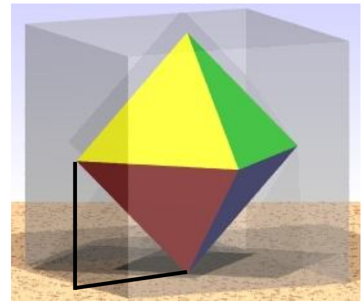
5. Using distinct digits from the set {0, 1, 2, 3, 4, 7, 8}, how many even integers, greater than 200 but less than 1000, can be written?

$\boxed{85}$.

6. One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. By how many square units has the surface area of the rectangular solid decreased?

$\boxed{2}$.

1. *Each edge of the octahedron is the hypotenuse of an isosceles right triangle with legs measuring* $\frac{1}{2}$*, so the edge length is* $\frac{\sqrt{2}}{2}$*. The octahedron itself is essentially two square pyramids, each with height* $\frac{1}{2}$*, joined at their bases.*



$V_{pyramid} = \frac{1}{3}Bh = \frac{1}{3}s^2h = \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{1}{2}\right) = \frac{2}{24} = \frac{1}{12}$*, so the area of the octahedron is twice that, or* $\boxed{\frac{1}{6}}$*.*

2. *(Certainly this could be solved by using Heron's formula and then finding the maximum value of the quartic polynomial that ends up under the radical, but there* __*must*__ *be an easier way...)*

   Consider the angle between the sides of lengths 6 and 7 (call it α*). Then* $Area = \frac{1}{2}(6)(7)\sin\alpha$*. This will be a maximum when* $\sin\alpha$ *is a maximum, which occurs at* $\alpha = 90°$*. This creates a right triangle with legs 6 and 7, which has area* $\frac{1}{2}(6)(7) = \boxed{21}$ *.*

3. *The absolute difference between the sums of any two rows or columns will be 0, 2d, 4d, or 6d, depending on the number of times the entries in those rows were increased/decreased by d. (For example, if in one row the entries were all increased while in another they were all decreased, the rows' sums will differ by 6d.) The sums of the three rows are 162, 210, and 162, showing differences of 0 and 48. So 48 = 2d, 4d, or 6d. The sums of the three columns are 186, 186, and 162, showing differences of 0 and 24. So 24 = 2d, 4d, or 6d. The two scenarios are possible simultaneously only when 4d = 48 and 2d = 24, so d =* $\boxed{12}$.

4. *The n-th expression in the sequence consists of the sum of n consecutive integers, ending with the n-th triangular number,* $\frac{n(n+1)}{2}$*, and beginning immediately* __*after*__ *the (n–1)st triangular number:* $\frac{(n-1)n}{2}+1$*. Using the formula for the sum of an arithmetic series,* $S_n = (a_1 + a_n)\cdot\frac{n}{2}$*,*

   $$Sum=\left(\frac{n(n+1)}{2}+\frac{(n-1)n}{2}+1\right)\cdot\frac{n}{2}=\left(\frac{n^2+n+n^2-n+2}{2}\right)\cdot\frac{n}{2}=(n^2+1)\cdot\frac{n}{2}=(100^2+1)\cdot 50=\boxed{500050}.$$

5. *Focus on the first digit of the three-digit even integer you are trying to create. If this first digit is odd, then it must be a 3 or a 7 (2 choices). The last digit must then be 0, 2, 4, or 8 (4 choices), and since you have now used two of the seven digits from the set, you have 5 choices left for the middle digit. (2)(4)(5) = 40.*

   If the first digit is even, then it must be 2, 4, or 8 (3 choices). This leaves only 3 of the even digits available as choices for the last digit, and again, 5 choices remaining for the middle digit. (3)(3)(5) = 45.

   There are 40 integers in the first case, and 45 integers in the second case, for a total of $\boxed{85}$ *integers.*

6. *Let x be the side length of the cube. Then the surface area of the cube was* $6x^2$*, and the surface area of the new solid is* $2x(x+1)+2x(x-1)+2(x+1)(x-1)=2x^2\cancel{+2x}+2x^2\cancel{-2x}+2x^2-2=6x^2-2$*, which is* $\boxed{2}$ *less than the surface area of the original cube.* ***(from AMC 12A, #5)***