Minnesota State High School Mathematics League

Individual Event

2009-10 Event 4A SOLUTIONS

1. Compute the value of $\sqrt[3]{3^5+3^5+3^5}$.

$$\sqrt[3]{3^5+3^5+3^5}=\sqrt[3]{3\left(3^5\right)}=\sqrt[3]{3^6}=3^{6/3}=3^2=9.$$

9 .

2. The expression $(x+y)^3-x(x+y)^2-y(x+y)^2$ can be simplified so that it is ***written as just a single term.*** Do so.

The three terms share a common factor of $(x+y)^2$. *Factoring,*

$$(x+y)^3-x(x+y)^2-y(x+y)^2=(x+y)^2\left[(x+y)-x-y\right]$$
$$=(x+y)^2[0]=0.$$

0 .

3. The function f is defined by $f(n)=3\cdot f(n-1)-f(n-2)$, where n is any positive integer. If $f(1)=1$, and $f(2)=\frac{1}{3}$, evaluate $f(7)$.

***f* (7) =** **−7** .

$f(3)=3\cdot f(2)-f(1)=3\left(\frac{1}{3}\right)-1=0;\quad f(4)=3\cdot f(3)-f(2)=3(0)-\frac{1}{3}=-\frac{1}{3};$

$f(5)=3\left(-\frac{1}{3}\right)-0=-1;\quad f(6)=3(-1)-\left(-\frac{1}{3}\right)=-\frac{8}{3};\quad f(7)=3\left(-\frac{8}{3}\right)-(-1)=-7.$

4. In the equation $\dfrac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}=\dfrac{\sqrt{x}+\sqrt{y}}{2}$, both x and y are nonnegative integers.

Compute the sum $x+y$.

The left side can be rationalized in a couple of ways. One possibility is:

$$\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}\cdot\left(\frac{1+\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}\right)=\frac{1-\left(\sqrt{2}-\sqrt{3}\right)}{1+\left(\sqrt{2}-\sqrt{3}\right)}\cdot\frac{1+\left(\sqrt{2}+\sqrt{3}\right)}{1+\left(\sqrt{2}+\sqrt{3}\right)}$$

$$=\frac{1+\left(\sqrt{2}+\sqrt{3}\right)-\left(\sqrt{2}-\sqrt{3}\right)-(2-3)}{1+\left(\sqrt{2}+\sqrt{3}\right)+\left(\sqrt{2}-\sqrt{3}\right)+(2-3)}=\frac{1+2\sqrt{3}-(-1)}{1+2\sqrt{2}+(-1)}$$

$$=\frac{2+2\sqrt{3}}{2\sqrt{2}}=\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{2}+\sqrt{6}}{2}.$$

So $x=2$ and $y=6$ (or vice versa), and $x+y=8$.

8 .

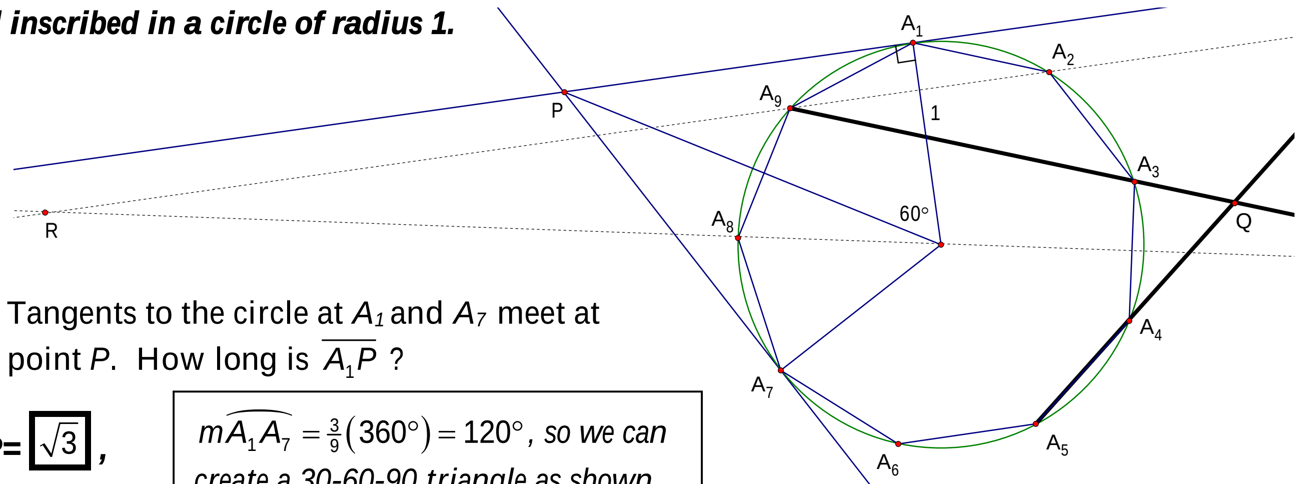
(Problems for this event are variations of those presented at the 1984 Coaches' Institute.)

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 4B
SOLUTIONS

Questions #1-3 all refer to a nine-sided regular polygon that is labeled $A_1 A_2 \ldots A_9$ and inscribed in a circle of radius 1.



1. Tangents to the circle at A_1 and A_7 meet at point P. How long is $\overline{A_1P}$?

A_1P= $\boxed{\sqrt{3}}$, **or** $\boxed{\approx 1.732}$.

$m\widehat{A_1A_7} = \frac{3}{9}(360°) = 120°$, so we can create a 30-60-90 triangle as shown. Radius = 1 $\Rightarrow$ $A_1P = \sqrt{3}$.

2. Secants containing $\overline{A_3A_9}$ and $\overline{A_4A_5}$ meet at point Q. What (***in degrees***) is the measure of $\angle A_5QA_9$?

$m\angle A_5QA_9$ = $\boxed{60°}$.

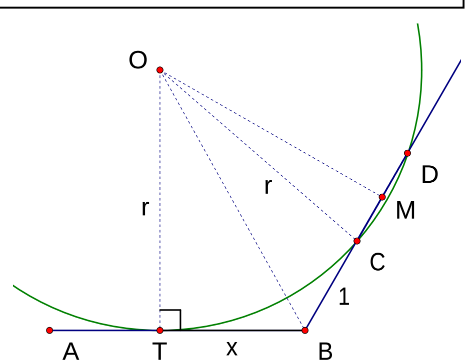
$\angle A_5QA_9$ subtends arcs $\widehat{A_5A_9}$ and $\widehat{A_3A_4}$, which have measures of 160º and 40º, respectively. By theorem, $m\angle A_5QA_9 = \frac{1}{2}\left(m\widehat{A_5A_9} - m\widehat{A_3A_4}\right) = \frac{1}{2}(160° - 40°) = 60°$.

3. The secant containing $\overline{A_2A_9}$ meets the extension of the diameter containing A_8 at point R. What (***in degrees***) is the measure of $\angle A_9RA_8$?

$m\angle A_5RA_8$ = $\boxed{10°}$.

$\angle A_9RA_8$ subtends the arc from A_2 to halfway between A_3 and A_4 (60º), and $\widehat{A_8A_9}$ (40º). By theorem, $m\angle A_9RA_8 = \frac{1}{2}(60° - 40°) = 10°$.

4. In *Figure 4*, $m\angle ABD = 120°$ and $BC = CD = 1$. A circle is drawn through C and D, tangent to $\overline{AB}$ at T. What will be the length of BT ?



$\boxed{\sqrt{2}}$, **or** $\boxed{\approx 1.414}$.

The $120°$ angle isn't actually needed. Using $\triangle TOB$, $OB = \sqrt{x^2 + r^2}$. Using $\triangle COM$ (with $CM = \frac{1}{2}$), $OM = \sqrt{r^2 - \frac{1}{4}}$. Using $\triangle BOM$ (with $BM = \frac{3}{2}$), $\left(\frac{3}{2}\right)^2 + \left(\sqrt{r^2 - \frac{1}{4}}\right)^2 = \left(\sqrt{x^2 + r^2}\right)^2 \Rightarrow x = \sqrt{2}$.

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 4C
SOLUTIONS

1. What is the 2010th positive odd number?

4019 .

The nth positive odd number will always be 1 less than the nth positive even number (2n). So we calculate: $2n-1=2(2010)-1=4020-1=4019$.

2. Find the sum of the infinite geometric series whose first two terms are 6! and 5! .

864 .

Since $(6!)\left(\frac{1}{6}\right)=5!$, *the common ratio must be* $\frac{1}{6}$. *Using the formula for the sum of an infinite converging geometric series,* $\dfrac{a_1}{1-r}=\dfrac{6!}{1-\left(\frac{1}{6}\right)}=\dfrac{6!}{\left(\frac{5}{6}\right)}=(6!)\left(\dfrac{6}{5}\right)=720\left(\dfrac{6}{5}\right)=864$.

3. The sum of the first ten terms of an arithmetic sequence is four times the sum of the first five terms. If the first term of the sequence is a_1 and the common difference is *d*, ***compute the ratio*** $a_1 : d$.

a_1 : d = **1 : 2** .

(Mathematics Teacher, November 2003)

Sum of first 10 terms: $a_1+(a_1+d)+(a_1+2d)+\ldots+(a_1+9d)=10a_1+45d$

Sum of first 5 terms: $a_1+(a_1+d)+(a_1+2d)+\ldots+(a_1+4d)=5a_1+10d$

So $10a_1+45d=4(5a_1+10d)=20a_1+40d \Rightarrow 5d=10a_1 \Rightarrow a_1:d=1:2$.

4. If $f(x)=1-\dfrac{1}{x}$, ***find the exact value*** of *x* for which $\underbrace{f\left(f\left(f\left(f\left(f\left(f\left(f\ldots\left(f(x)\right)\right)\right)\right)\right)\right)\right)}_{\text{2009 applications of the function } f}=2010$.

x = $\boxed{\dfrac{2009}{2010}}$.

Maybe it would be easier to start with just a <u>*few*</u> *applications of f:*

$$f(f(x))=1-\frac{1}{\left(1-\frac{1}{x}\right)}=1-\frac{1}{\left(\frac{x-1}{x}\right)}=1-\frac{x}{x-1}=\frac{x-1-x}{x-1}=\frac{-1}{x-1}=\frac{1}{1-x};$$

$$f(f(f(x)))=f\left(\frac{1}{1-x}\right)=1-\frac{1}{\left(\frac{1}{1-x}\right)}=1-(1-x)=x.$$ *Eureka! Every three applications of the function is a "reset"; that is, it returns the original input. If there are 2009 applications of f, we simply note that 2009 is 2 more than a multiple of 3. So*

$$\underbrace{f\left(f\left(f\left(f\left(f\left(f\left(f\ldots\left(f(x)\right)\right)\right)\right)\right)\right)\right)}_{\text{2009 applications of the function } f}=f(f(x))=\frac{1}{1-x}=2010.$$ *Solving,* $x=\dfrac{2009}{2010}$.

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 4D SOLUTIONS

1. Give the coordinates of the center of the circle described by $x^2 + y^2 - 12x + 10y - 38 = 0$.

(h, k) = $\boxed{(6, -5)}$ **.**

It is sufficient to examine the x and y terms to see that in the process of completing the square, the terms $(x-6)^2$ *and* $(y+5)^2$ *will be created.*

2. Compute the area of the circle described by $x^2 + y^2 + 2x + 6y + 3 = 0$.

$\boxed{7\pi}$ ***,***

or $\boxed{\approx 21.991}$.

Completing the square, $(x^2 + 2x + 1) + (y^2 + 6y + 9) + 3 = 1 + 9$.

$\Rightarrow\ (x+1)^2 + (y+3)^2 = 7$. *Since the radius squared is 7, Area =* $\pi r^2 = \pi(7)$.

3. The asymptotes of a hyperbola are the lines $y = 2x$ and $y = -2x$. If the hyperbola passes through the point (9, 16), find ***the x-coordinate of the hyperbola's <u>positive</u> x-intercept.***

x = $\boxed{\sqrt{17}}$ ***,***

or $\boxed{\approx 4.123}$.

The center of the hyperbola must be at the origin (where the asymptotes intersect), and since (9, 16) is located to the <u>right</u> of (8, 16), which lies on an asymptote, the hyperbola's branches open horizontally, with equation $\frac{x^2}{a^2} - \frac{y^2}{(2a)^2} = 1$. *This rearranges to* $x^2 - \frac{y^2}{4} = a^2$. *Substituting (9, 16) yields* $a^2 = 17$, *so* $x = +\sqrt{17}$.

4. A hyperbola has its foci on the x-axis and passes through the points (–1, 0), (2, 0), and (–2, 1). Compute ***the x-coordinate of the <u>right-most</u> focus.***

x = $\boxed{\frac{1}{2} + \frac{3\sqrt{5}}{4}}$ ***,***

or $\boxed{\approx 2.177}$.

Since (–1, 0) and (2, 0) are on the x-axis, they must be the vertices of the hyperbola. Furthermore, the center is located at $\left(\frac{1}{2}, 0\right)$ *and the hyperbola opens horizontally. So far, this gives us* $\frac{\left(x - \frac{1}{2}\right)^2}{\left(\frac{3}{2}\right)^2} - \frac{y^2}{b^2} = 1$. *The only point we haven't used is (–2, 1), so substitute these coordinates for x and y:*

$$\frac{\left(-2 - \frac{1}{2}\right)^2}{\left(\frac{3}{2}\right)^2} - \frac{1^2}{b^2} = 1 \ \Rightarrow\ \frac{\left(\frac{25}{4}\right)}{\left(\frac{9}{4}\right)} - \frac{1}{b^2} = 1 \ \Rightarrow\ \frac{25}{9} - 1 = \frac{1}{b^2} \ \Rightarrow\ b^2 = \frac{9}{16} \ \Rightarrow\ b = \frac{3}{4}.$$

Now use the hyperbola identity: $c^2 = a^2 + b^2 = \frac{9}{4} + \left(\frac{3}{4}\right)^2 = \frac{45}{16} \ \Rightarrow\ c = \frac{3\sqrt{5}}{4}$.

The right-most focus is located c units right of the center, at x = $\frac{1}{2} + \frac{3\sqrt{5}}{4}$.

Minnesota State High School Mathematics League

Team Event

2009-10 Meet 4
SOLUTIONS

1. A triangle inscribed in a circle has side lengths 12, $12\sqrt{2}$, and $6\sqrt{6}+6\sqrt{2}$. Compute the length of the circle's diameter.

$\boxed{24}$.

2. What is the least positive integer *n* > 1 for which the expression $\sqrt{1+2+3+...+n}$ simplifies to an integer? ***(Mathematics Teacher, Oct. 2006)***

n **=** $\boxed{8}$ **.**

3. The expression $\dfrac{\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{5}+\sqrt{2}\right)}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ can be rationalized into ***a single fraction whose denominator is a positive integer.*** Do so.

$\boxed{\dfrac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}}$.

4. Given $f(\theta)=(1+\cos\theta)\sqrt{\dfrac{\sec\theta-1}{\sec\theta+1}}$, express $f(0)+f\left(\dfrac{\pi}{6}\right)+f\left(\dfrac{\pi}{4}\right)+f\left(\dfrac{\pi}{3}\right)$ accurate to three places to the right of the decimal.

$\boxed{\dfrac{1+\sqrt{2}+\sqrt{3}}{2}}$ ***, or*** $\boxed{2.073}$ **.**

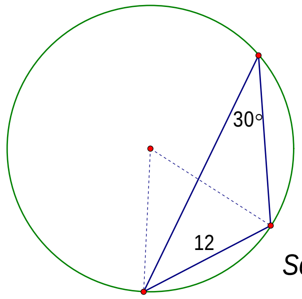
5. When the radical equation $\sqrt{x+1-2\sqrt{x}}=15$ is solved using the typical method of squaring both sides repeatedly, two solutions are discovered for *x*, but one of these solutions is extraneous. Find ***the value of that <u>extraneous</u> solution.***

$\boxed{196}$ **.**

6. A parabola is defined as the curve containing all points equidistant from a focus *F* and a line called the directrix. Let us define a *quasi-parabola* to be the curve containing all points equidistant from *F* and a <u>line segment</u> called the directrix***. Find all x- and y-intercepts*** of the quasi-parabola with *F* = (9, 9) and directrix with endpoints *A* = (3, 7) and *B* = (7, 5).

$\boxed{x-\text{int}=(22,0);\ y-\text{int}=(0,26)}$. ***+2 points for each correct intercept***

1. *Using the Law of Cosines,*



$$12^2 = \left(12\sqrt{2}\right)^2 + \left(6\sqrt{6}+6\sqrt{2}\right)^2 - 2\left(12\sqrt{2}\right)\left(6\sqrt{6}+6\sqrt{2}\right)\cos\alpha$$
$$144 = 288 + \left(216 + 72\sqrt{12} + 72\right) - \left(144\sqrt{12} + 288\right)\cos\alpha$$
$$144 = 576 + 144\sqrt{3} - \left(288\sqrt{3} + 288\right)\cos\alpha$$
$$\frac{-432-144\sqrt{3}}{-288-288\sqrt{3}} = \frac{3+\sqrt{3}}{2+2\sqrt{3}}\left(\frac{2-2\sqrt{3}}{2-2\sqrt{3}}\right) = \frac{-4\sqrt{3}}{-8} = \frac{\sqrt{3}}{2} = \cos\alpha$$

So $\alpha = 30°$, and since α is an exterior angle, there is an equilateral triangle of side length 12 at the circle's center. Radius = 12 $\Rightarrow$ Diameter = $\boxed{24}$.

2. *Using change of base, $\sqrt{1+2+3+...+n} = \sqrt{\frac{n(n+1)}{2}}$. Suppose this simplifies to some integer k. Then $\frac{n(n+1)}{2} = k^2 \Rightarrow n(n+1) = 2k^2$. We are looking for two consecutive integers whose product is double a perfect square...(see table)*

So the first n > 1 that works is n = $\boxed{8}$.

n(n+1)	*2(3)*	*3(4)*	*4(5)*	*5(6)*	*6(7)*	*7(8)*	*8(9)*
Cut in half	*3*	*6*	*10*	*15*	*21*	*28*	***36***

3. *Let x = $\sqrt{2}+\sqrt{3}+\sqrt{5}$. Then the expression can be written:*

$$\frac{\left(x-\sqrt{2}\right)\left(x-\sqrt{3}\right)}{x} = \frac{x^2-\left(\sqrt{2}+\sqrt{3}\right)x+\sqrt{6}}{x} = x-\left(\sqrt{2}+\sqrt{3}\right)+\frac{\sqrt{6}}{x} = \sqrt{5}+\frac{\sqrt{6}}{x}.$$

Now rationalize: $\sqrt{5}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}\cdot\dfrac{\sqrt{2}+\left(\sqrt{3}-\sqrt{5}\right)}{\sqrt{2}+\left(\sqrt{3}-\sqrt{5}\right)} = \sqrt{5}+\dfrac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{2+\sqrt{6}-\cancel{\sqrt{10}}+\sqrt{6}+\cancel{\sqrt{10}}+(3-5)}$

$$= \sqrt{5}+\frac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{2\sqrt{6}} = \sqrt{5}+\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2} = \boxed{\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}}.$$

4. *First note that* $\sqrt{\dfrac{\sec\theta-1}{\sec\theta+1}} = \sqrt{\dfrac{\sec\theta-1}{\sec\theta+1}\left(\dfrac{\sec\theta-1}{\sec\theta-1}\right)} = \sqrt{\dfrac{(\sec\theta-1)^2}{\sec^2\theta-1}} = \dfrac{\sec\theta-1}{\tan\theta}$. *Then multiply top and bottom of by* $\cos\theta$ *to obtain* $\dfrac{1-\cos\theta}{\sin\theta}$. *So* $f(\theta) = (1+\cos\theta)\left(\dfrac{1-\cos\theta}{\sin\theta}\right) = \dfrac{1-\cos^2\theta}{\sin\theta} = \dfrac{\sin^2\theta}{\sin\theta} = \sin\theta$,

and $f(0)+f\left(\dfrac{\pi}{6}\right)+f\left(\dfrac{\pi}{4}\right)+f\left(\dfrac{\pi}{3}\right) = 0+\dfrac{1}{2}+\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{3}}{2} = \boxed{\dfrac{1+\sqrt{2}+\sqrt{3}}{2}}$.

5. *Squaring both sides yields:* $\left(\sqrt{x+1-2\sqrt{x}}\right)^2 = (15)^2 \Rightarrow x+1-2\sqrt{x} = 225 \Rightarrow 2\sqrt{x} = x-224$.

Squaring both sides a second time yields: $4x = x^2-448x+50176 \Rightarrow x^2-452x+50176.$

Solving this quadratic reveals roots of 196 and 256. 256 works in the original equation; $\boxed{196}$ *doesn't.*

6. *A quasi-parabola looks like a section of a parabola joined to two rays. In this case, the parabola section comes nowhere near the axes, so we need only concern ourselves with the two rays. These rays lie on the perpendicular bisectors of $\overline{AF}$ and $\overline{BF}$. One ray emanates from (6,8) with slope –3:* $y-8 = -3(x-6)$, *where* $x \le 6$. *This has only a y-intercept, at* $\boxed{(0,26)}$. *The other ray emanates from (8, 7) with slope* $-\frac{1}{2}$: $y-7 = -\frac{1}{2}(x-8)$, *where* $x \ge 8$. *This has only an x-intercept, at* $\boxed{(22,0)}$.