Minnesota State High School Mathematics League

Individual Event

2009-10 Event 3A
SOLUTIONS

1. As the value of *k* changes, the equation $(2x-3y+6)+k(2x+y-10)=0$ describes a family of lines. What value of *k* identifies the vertical line in this family?

***k* =** 3 **.**

Setting k = 3 causes the y-terms to drop out, leaving an equation involving only x.

2. Solve the system: $\begin{cases} 2x-3y+6=0 \\ 2x+y-10=0 \end{cases}$

(x, y) = $(3,4)$.

$\Rightarrow \begin{cases} 2x=3y-6 \\ 2x=10-y \end{cases}$ *Equating the right sides,*

$3y-6=10-y \;\Rightarrow\; 4y=16 \;\Rightarrow\; y=4$.

Substituting, $2x=3(4)-6 \;\Rightarrow\; x=3$.

3. The system $\begin{cases} p + q + 2r = 1 \\ 2p - q = 3 \\ -p + 4r = 1 \end{cases}$ has a unique solution. Find it.

(p, q, r) = $\left(1, -1, \frac{1}{2}\right)$.

Adding the first two equations and relating the result to the third equation:

$\begin{cases} 3p+2r=4 \\ -p+4r=1 \end{cases}$ *Now triple the bottom equation and add:* $14r=7 \;\Rightarrow\; r=\frac{1}{2}$.

Substituting, $-p+4\left(\frac{1}{2}\right)=1 \;\Rightarrow\; p=1$, *and again,* $2(1)-q=3 \;\Rightarrow\; q=-1$.

4. Consider all possible ordered triples of integers (*x, y, z*) which solve the system $\begin{cases} x+3y-z=1 \\ 3x-y-2z=-1 \end{cases}$. There are two specific single-digit positive integers, *n* and *d*, such that all of the *x*-values in those ordered triples can be written in the form *mn* + *d*, where *m* is any integer. Compute the values of *n* and *d*.

***n* =** 7 **,**

***d* =** 4 **.**

Multiply the top equation by –2 and add: $x-7y=-3 \;\Rightarrow\; y=\frac{x+3}{7}$.

Then multiply the bottom equation by 3 and add: $10x-7z=-2 \;\Rightarrow\; z=\frac{10x+2}{7}$.

We need x + 3 and 10x + 2 to both be multiples of 7. This occurs when x = 4, and every 7 units thereafter. So y and z will be integers when x is of the form 7m + 4.

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 3B SOLUTIONS

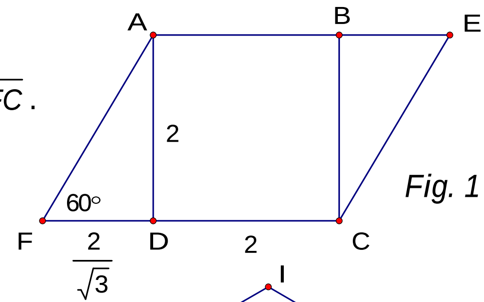


Fig. 1

1. *Figure 1* shows square *ABCD*, of side length 2, inscribed in parallelogram *AECF*. If $m\angle F = 60°$, compute the length of $\overline{FC}$.

FC = $\boxed{2+\frac{2}{\sqrt{3}}}$, **or** $\boxed{\approx 3.155}$.

DC = 2. $\triangle ADF$ is 30-60-90, so $FD = \frac{AD}{\sqrt{3}} = \frac{2}{\sqrt{3}}$. FC = FD+DC.

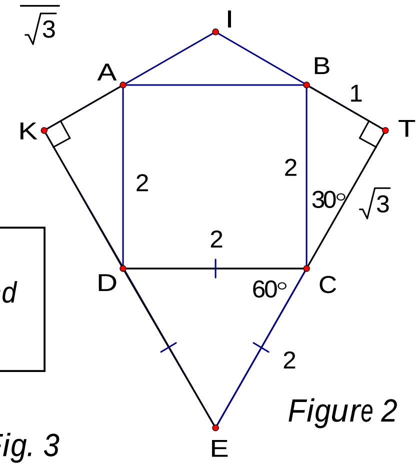


Figure 2

2. *Figure 2* shows square *ABCD*, of side length 2, inscribed in kite *KITE* so that $\triangle DCE$ is equilateral. If angles *K* and *T* are right angles, compute the length of $\overline{ET}$.

ET = $\boxed{2+\sqrt{3}}$, **or** $\boxed{\approx 3.732}$.

Chase angles around point C to find that $m\angle BCT = 30°$. So $\triangle BTC$ is 30-60-90, BT = 1, and $CT = \sqrt{3}$. ET = EC + CT = $2+\sqrt{3}$.

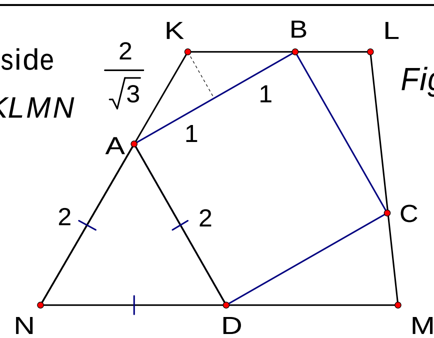


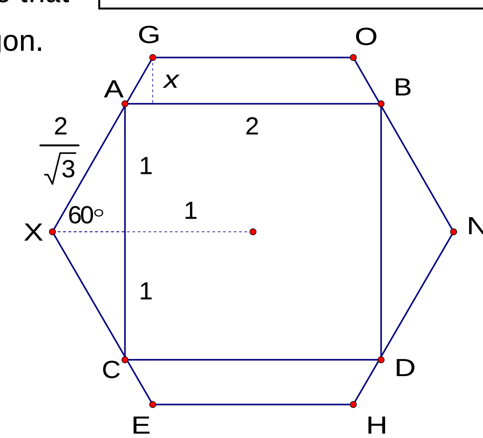
Fig. 3

3. *Figure 3* shows square *ABCD*, of side length 2, inscribed in trapezoid *KLMN* so that $\triangle AND$ is equilateral. Compute the length of $\overline{NK}$.

NK = $\boxed{2+\frac{2}{\sqrt{3}}}$, **or** $\boxed{\approx 3.155}$.

Chase angles around point A to find that $m\angle KAB = 30°$. Drop an altitude from K to $\overline{AB}$, forming a 30-60-90 triangle. So $AK = \frac{2}{\sqrt{3}}$, and NK = NA + AK = $2+\frac{2}{\sqrt{3}}$.

4. A square of side length 2 is inscribed in a regular hexagon so that two sides of the square are parallel to two sides of the hexagon. Find the shortest distance between two of the parallel sides.



$\boxed{\frac{\sqrt{3}-1}{2}}$, **or** $\boxed{\approx 0.366}$.

Drop altitudes from X and G to create similar 30-60-90 triangles, with heights x and 1. Using side ratios, $AX = \frac{2}{\sqrt{3}}$. Using similar triangles, $\frac{x}{1} = \frac{AG}{AX} \Rightarrow AG = \frac{2}{\sqrt{3}}x$. But XG has the same length as the hexagon's radius $\left(1+\frac{1}{\sqrt{3}}\right)$. Solving $\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}}x = 1 + \frac{1}{\sqrt{3}}$, $x = \frac{\sqrt{3}-1}{2}$.

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 3C SOLUTIONS

1. Compute the value of $\text{Sin}^{-1}\left(\frac{1}{2}\right)+\text{Cos}^{-1}\left(\frac{-1}{2}\right)$.

$\boxed{\frac{5\pi}{6}}$, ***or*** $\boxed{150^\circ}$.

$$\text{Sin}^{-1}\left(\frac{1}{2}\right)+\text{Cos}^{-1}\left(\frac{-1}{2}\right)=\frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6}.$$

2. Express the range of the function $f(x)=\cos\left(\text{Tan}^{-1}x\right)$. Do this by placing inequality symbols in the boxes, and real numbers, $-\infty$, or $+\infty$ in the blanks.

$\boxed{0<f(x)\le 1}$.

The definition of the inverse tangent function is $-\frac{\pi}{2}<\text{Tan}^{-1}x<\frac{\pi}{2}$, *so we are looking for the values cosine can assume in quadrants I and IV and along the positive x-axis. These values are found on the interval (0, 1].*

3. Right triangle *ABC* (shown in *Figure 3*) has legs of lengths $AB=4$ and $AC=6$, and point D is the midpoint of side $\overline{AC}$. Compute $\sin\angle CBD$.

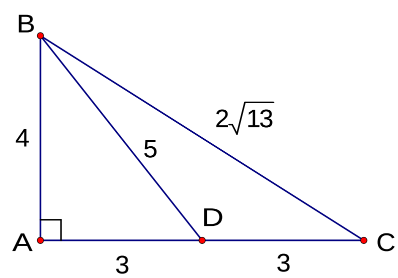


Figure 3

$\boxed{\frac{6\sqrt{13}}{65}}$, ***or*** $\boxed{\approx 0.333}$.

$\triangle ABD$ *is 3-4-5, and by Pythagoras,* $BC=2\sqrt{13}$. *Using* $\triangle ABC$, $\sin\angle C=\frac{2}{\sqrt{13}}$. *By the Law of Sines,*

$$\frac{\sin C}{5}=\frac{\sin\angle CBD}{3}\Rightarrow \sin\angle CBD=\frac{6}{5\sqrt{13}}.$$

4. The side lengths of a certain triangle are three consecutive integers, and the smallest angle in the triangle has a tangent of $\frac{2\sqrt{6}}{5}$. Find the exact value of the triangle's perimeter.

$\boxed{18}$.

Label the three side lengths as $a-1$, *a, and* $a+1$. *The shortest side,* $a-1$, *must be opposite the smallest angle, so by the Law of Cosines:*

$$(a-1)^2=a^2+(a+1)^2-2a(a+1)\cos\left(\arctan\frac{2\sqrt{6}}{5}\right)\Rightarrow\frac{5}{7}=\frac{a^2-2a+1-\left(a^2+a^2+2a+1\right)}{-2a(a+1)}$$

$$\Rightarrow\frac{5}{7}=\frac{-a^2-4a}{-2a(a+1)}=\frac{a+4}{2(a+1)}\Rightarrow 7a+28=10a+10\Rightarrow a=6,\ \textit{and } P=3a=18.$$

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 3D SOLUTIONS

1. If $\log_4 M = \frac{5}{2}$, then M can be written in the form 2^n. Compute n.

***n* = 5.**

$$\log_4 M = \frac{5}{2} \;\Rightarrow\; M = 4^{5/2} = \left(\sqrt{4}\right)^5 = 2^5.$$

2. Compute the value of the sum: $\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \ldots + \log\frac{n}{n+1} + \ldots + \log\frac{99}{100}$.

−2.

(NYC Contest Problem Book, © 1986)

$$\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \ldots + \log\frac{n}{n+1} + \ldots + \log\frac{99}{100}$$
$$= (\log 1 - \log 2) + (\log 2 - \log 3) + (\log 3 - \log 4) + \ldots + (\log 99 - \log 100)$$
$$= \log 1 - \log 100 = 0 - 2 = -2.$$

(1st HS Math League Problem Book, © 1989)

3. If $b = \log_3 x$, find all real values of x which satisfy $\log_b\left(\log_3 x^2\right) = 2$.

x* = 9.** ***1 point for incorrectly listing ±9; -9 is extraneous because of the definition of b.

$$\log_b\left(\log_3 x^2\right) = 2 \;\Rightarrow\; \log_b\left(2\cdot\log_3 x\right) = 2 \;\Rightarrow\; \log_b(2b) = 2$$
$$\Rightarrow\; \log_b 2 + \log_b b = \log_b 2 + 1 = 2 \;\Rightarrow\; \log_b 2 = 1, \text{ so } b = 2.$$

Since $b = \log_3 x$, *x must equal 9.*

4. If log 80 = *a* and log 45 = *b*, write log 6 as a simplified expression involving *a* and *b*.

log 6 = $\boxed{\dfrac{a+b}{2} - 1}$**.**

(This hints that log 60 is exactly the average of a and b. How does 60 relate, numerically, to 45 and 80? Hmm...)

$$a + b = \log 80 + \log 45$$
$$= \log(80\cdot 45)$$
$$= \log 3600 = \log(36\cdot 100) = \log 36 + \log 100$$
$$= \log 36 + 2.$$

So $\log 36 = \log 6^2 = 2\log 6 = a + b - 2$, *and* $\log 6 = \dfrac{a+b-2}{2}$.

Minnesota State High School Mathematics League

Team Event

2009-10 Meet 3
SOLUTIONS

1. Find the value of n such that $\frac{1}{1+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+...+\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}=100$.

***n* =** $\boxed{20200}$. *(Mathematics Teacher, August 2006)*

2. The sides of a triangle have lengths $\log_8 a$, $\log_{16} a$, and $\log_{64} a$ for some value of a. Compute the sine of the smallest angle in this triangle.

$\boxed{\frac{\sqrt{15}}{8}}$***, or*** $\boxed{\approx 0.484}$**.**

3. *Figure 3* shows square *ABCD*, of side length 2, inscribed in trapezoid *KLMN* so that $\triangle AND$ is equilateral. If $m\angle M=\alpha$, express the length of $\overline{LM}$ in terms of $\sin\alpha$.

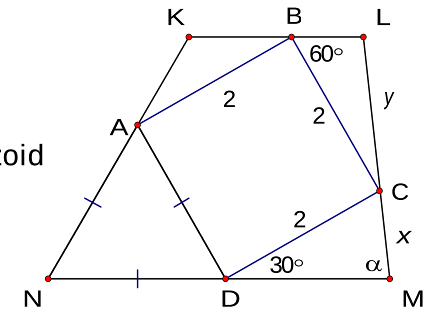


Figure 3

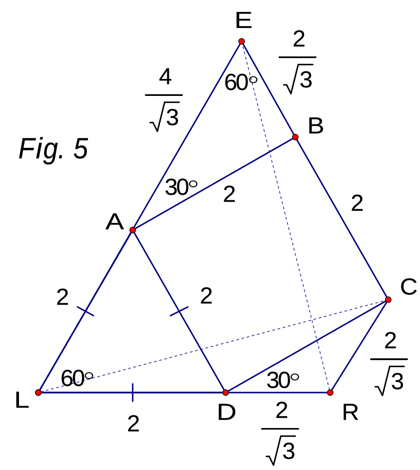
***LM* =** $\boxed{\frac{1+\sqrt{3}}{\sin\alpha}}$.

4. Two people, both starting at sunrise, walk toward each other from opposite ends of a hiking trail. They meet at noon. Both continue hiking, the faster one finishing the trail at 5:20 pm, the slower one finishing at 8:20 pm. What time was sunrise?

$\boxed{5:20 \text{ am}}$.

5. *Figure 5* shows square *ABCD*, of side length 2, inscribed in cyclic quadrilateral *CRLE* so that $\triangle ALD$ is equilateral. Compute the length of diagonal $\overline{LC}$.

Fig. 5



LC* =** $\boxed{\sqrt{2}+\sqrt{6}}$, or*** $\boxed{\approx 3.864}$**.**

6. Alice, Beth, and Cindy, starting together, walk in the same direction around a circular track. It takes the girls $5/36$, $2/9$, and $35/99$ of an hour, respectively, to walk once around the track. How many laps will Alice have walked by the next time all three girls are together again at the starting point?

$\boxed{56 \text{ laps}}$.

1. *Multiply the top and bottom of each fractional term by the conjugate of its denominator:*

$$\frac{1}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}\cdot\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}+\ldots+\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\cdot\frac{\sqrt{2n-1}-\sqrt{2n+1}}{\sqrt{2n-1}-\sqrt{2n+1}}=100$$

$$\frac{1-\sqrt{3}}{1-3}+\frac{\sqrt{3}-\sqrt{5}}{3-5}+\frac{\sqrt{5}-\sqrt{7}}{5-7}\ldots+\frac{\sqrt{2n-1}-\sqrt{2n+1}}{(2n-1)-(2n+1)}=\frac{1-\sqrt{2n+1}}{-2}=100$$

So $\sqrt{2n+1}=201 \Rightarrow 2n+1=201^2 \Rightarrow 2n=(201-1)(201+1) \Rightarrow n=(200)(101)=\boxed{20200}$.

2. *Using change of base,* $\log_8 a=\frac{\log_2 a}{\log_2 8}=\frac{1}{3}\log_2 a$. *Similarly,* $\log_{16} a=\frac{1}{4}\log_2 a$, *and* $\log_{64} a=\frac{1}{6}\log_2 a$. *So this triangle is similar to one with side lengths* $\frac{1}{3}$, $\frac{1}{4}$, *and* $\frac{1}{6}$. *Multiplying by a scale factor of the LCD (12) produces a triangle with integer side lengths: 4, 3, 2. Call the smallest angle* $\angle S$. *By the Law of Cosines,* $2^2=3^2+4^2-2(3)(4)\cos S \Rightarrow \cos S=\frac{4-9-16}{-24}=\frac{7}{8}$, *and* $\sin S=\sqrt{1-\cos^2 S}=\boxed{\frac{\sqrt{15}}{8}}$.

3. *Chasing angles,* $m\angle CDM=30°$ *and* $m\angle CBL=60°$. *Label CM = x and CL = y. By the Law of Sines,* $\frac{\sin 30°}{x}=\frac{\sin\alpha}{2} \Rightarrow x\sin\alpha=1 \Rightarrow x=\frac{1}{\sin\alpha}$, *and* $\frac{\sin 60°}{y}=\frac{\sin(180°-\alpha)}{2} \Rightarrow y\sin\alpha=\sqrt{3}$
$\Rightarrow y=\frac{\sqrt{3}}{\sin\alpha}$. *So LM = CM + CL = x + y =* $\boxed{\frac{1+\sqrt{3}}{\sin\alpha}}$.

4. *Let s be the speed of the slow walker, f the speed of the fast walker, and t the number of hours between sunrise and noon. The distance that the fast walker travels before noon is the same distance that the slow walker travels after noon, so* $ft=\frac{25}{3}s$. *Also, the distance that the slow walker travels before noon is the same distance that the fast walker travels after noon, so* $st=\frac{16}{3}f$. *Multiplying these two equations yields* $fst^2=\frac{25}{3}\cdot\frac{16}{3}fs \Rightarrow t^2=\frac{25\cdot16}{9} \Rightarrow t=\frac{5\cdot4}{3}=6\frac{2}{3}$ hours. *Sunrise occurred at* $\boxed{5:20\text{ am}}$.

5. *Chasing angles,* $m\angle EAB=30°$ *and* $m\angle CDR=30°$. *Various 30-60-90 relationships allow us to label the perimeter of CRLE as shown. Note that* $\Delta LER\cong\Delta ELC$, *so diagonals* $\overline{CL}$ *and* $\overline{ER}$ *are congruent! Because CRLE is cyclic, we apply Ptolemy's Theorem:* $EL\cdot CR+LR\cdot EC=LC\cdot ER$

$\Rightarrow \left(2+\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)+\left(2+\frac{2}{\sqrt{3}}\right)^2=\frac{4}{\sqrt{3}}+\frac{8}{3}+\left(4+\frac{8}{\sqrt{3}}+\frac{4}{3}\right)=\frac{12}{\sqrt{3}}+8=4\sqrt{3}+8=CL^2$. *Hoping* $4\sqrt{3}+8$ *is a perfect square, set* $CL=a+b\sqrt{3}$ *and solve the resulting system:* $a=\sqrt{2}$, $b=\sqrt{2} \Rightarrow CL=\boxed{\sqrt{2}+\sqrt{6}}$.

6. *First find the LCD of the three rates:* $36=2^2\cdot3^2$, $9=3^2$, $99=3^2\cdot11 \Rightarrow LCD=2^2\cdot3^2\cdot11=396$.

$r_A=\frac{55}{396}$, $r_B=\frac{88}{396}$, $r_C=\frac{140}{396}$. *Now examine the numerators:* $55=5\cdot11$, $88=2^3\cdot11$, $140=2^2\cdot5\cdot7$. *What factors is Alice's numerator missing?* $2^3\cdot7=\boxed{56\text{ laps}}$. ***(Why does this work?)***