Minnesota State High School Mathematics League

Individual Event

2009-10 Event 1A
SOLUTIONS

1. Express $\frac{\frac{2}{3}+\frac{1}{2}}{2}+\frac{1}{\frac{2}{3}+\frac{1}{2}}$ as the quotient of two relatively prime integers.

$\boxed{\frac{121}{84}}$.

$$\frac{6}{6}\left(\frac{\frac{2}{3}+\frac{1}{2}}{2}+\frac{1}{\frac{2}{3}+\frac{1}{2}}\right)=\frac{4+3}{12}+\frac{6}{4+3}=\frac{7}{12}+\frac{6}{7}=\frac{7\cdot 7+12\cdot 6}{12\cdot 7}=\frac{121}{84}.$$

Graders' note: <u>No</u> alternate forms accepted here!

2. Alec is hosting a large party and needs to buy 100 bottles of Sierra Dew. Luckily, CostCorp sells them in bulk! He can buy 36 bottles for \$9.00, 15 bottles for \$4.50, or 4 bottles for \$1.60. What's the smallest amount Alec could pay, if he were to buy exactly 100 bottles?

$\boxed{\$28.60}$.

36 bottles cost \$0.25 each, 15 bottles cost \$0.30 each, and 4 bottles cost \$0.40 each. Working based on the number of 36-bottle packs:
Two 36's can only reach 100 with seven 4's: 2(\$9) + 7(\$1.60) = \$29.20;
One 36 reaches 100 with four 15's and one 4: 1(\$9) + 4(\$4.50) + 1(\$1.60) =\$28.60; With no 36's, we maximize 15's: 4(\$4.50) + 10(\$1.60) = \$34.

3. Find integers *m* and *n*, with $m < n$, for which the least common multiple of *m* and *n* is 105 and $50 < m + n < 70$.

***m* =** $\boxed{21}$, ***n* =** $\boxed{35}$.

$105 = 3\cdot 5\cdot 7$. *The divisors of 105 are: 1, 3, 5, 7, 15, 21, 35, 105. Of these, 21 and 35 are the only pair whose sum is between 50 and 70.*

4. In order to cope with the recession in auto sales, Acana Mist keeps adjusting the price for which she is selling her used car, which is worth \$10,000. Starting at this price prior to day 1, each day Acana either raises or lowers the price by 20% (but never asks less than the car is worth). Eventually, after *d* days of adjusting the price, she finds a buyer and makes a small profit – less than 5% of the car's worth. What is the smallest possible value for *d*?

***d* =** $\boxed{9}$.

*The price after **a** increases and **b** decreases is* $(1.2)^a(0.8)^b$ *times \$10,000. We want to find values for **a** and **b** such that* $1<(1.2)^a(0.8)^b<1.05$, *and **a+b** is a minimum. Since* $(1.2)(0.8)=0.96$, *we know that there must be more price increases than decreases. Use a calculator:* $a=2,\ b=1 \Rightarrow (1.2)^a(0.8)^b=1.152$,
$a=3,\ b=2 \Rightarrow (1.2)^a(0.8)^b\approx 1.106$, $a=4,\ b=3 \Rightarrow (1.2)^a(0.8)^b\approx 1.062$,
$a=5,\ b=4 \Rightarrow (1.2)^a(0.8)^b\approx 1.019$. *So **a+b** = 9.*

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 1B
SOLUTIONS

1. In *Figure 1*, $\triangle ABC$ is an isosceles triangle with $AB = AC$. Find $m\angle DAC + m\angle BCF + m\angle CBE$.

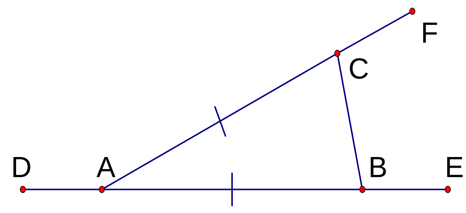


Figure 1

$\boxed{360°}$. *These are the exterior angles of the triangle! By the Exterior Angle Theorem, their sum is 360º.*

2. On hypotenuse $\overline{PR}$ of right triangle *PQR (Figure 2)*, *X* is the point for which *PX* = *PQ*. If $m\angle P = 40°$, find $m\angle RQX$.

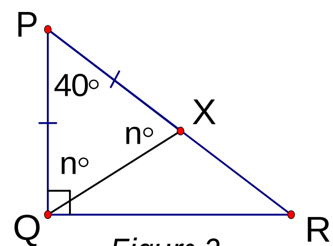


Figure 2

$\boxed{20°}$. *Observe the labeling in Figure 2.* $40° + n° + n° = 180°$, *so* $n = 70°$. *Then* $m\angle RQX = 90° - n° = 20°$.

3. In the regular 9-sided polygon shown in *Figure 3*, find the measure of the acute angle formed by the extensions of the two darkened sides.

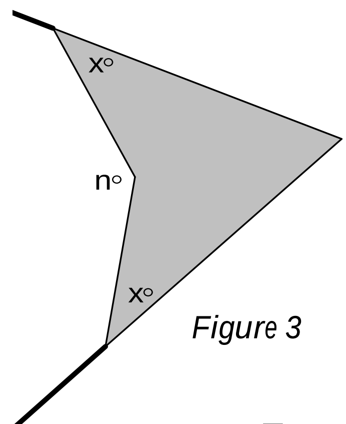


Figure 3

$\boxed{60°}$. *The concave quadrilateral created by the extended sides is shaded. Label the nonagon's exterior angles **x**, and its interior angles **n**.* $x = \dfrac{360°}{9} = 40°$, *and* $n = 180° - x = 140°$.
The desired angle is $360° - 2x - (360° - n) = n - 2x = 60°$.

4. In *Figure 4, AB = BC = CD = DE = EF = FG = GA*. Find $m\angle A$.

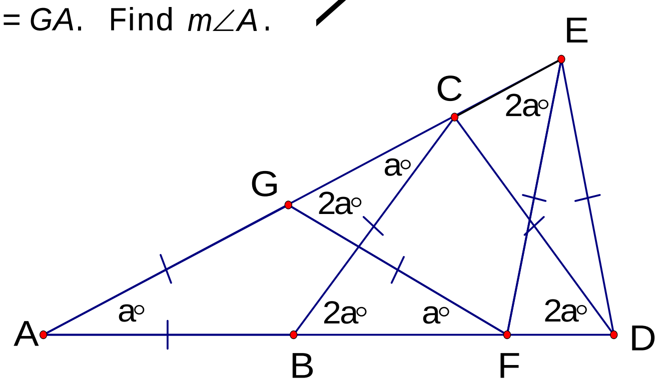


Figure 4

$\boxed{\dfrac{180°}{7}}$, ***or*** $\boxed{\approx 25.714°}$.

Label Figure 4 as shown, using isosceles triangles and exterior angles. Then, $m\angle BCD =$ $(180 - 4a)°$, *forcing* $\angle DCE$, $\angle CED$, $\angle EFD$, *and* $\angle EDF$ *to measure* $3a°$ *each. So* $m\angle FED = a°$, *and in* $\triangle FED$:

$$a + 3a + 3a = 180°$$
$$7a = 180° \Rightarrow a = (180/7)°$$

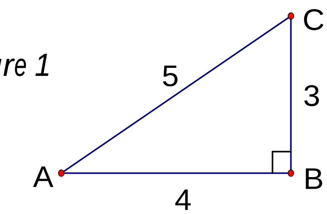
Minnesota State High School Mathematics League

Individual Event

2009-10 Event 1C SOLUTIONS

1. In *Figure 1*, $AB = 4$, $BC = 3$, and $AC = 5$. Find cot A.

Figure 1



$\boxed{\frac{4}{3}}$, **or** $\boxed{\approx 1.333}$.

$$\boxed{\cot A = \frac{\textit{adjacent leg}}{\textit{opposite leg}} = \frac{4}{3}.}$$

2. Again use *Figure 1*, but this time ignore the values from problem #1.

Now, $\sin A = \frac{3}{5}$ and $AC = 7$. Find the length of $\overline{BC}$.

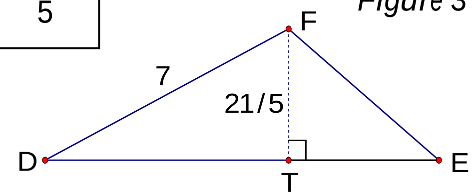
$\boxed{\frac{21}{5}}$, **or** $\boxed{4.2}$.

$$\boxed{\sin A = \frac{opp}{hyp} = \frac{BC}{7} = \frac{3}{5} \Rightarrow 5 \cdot BC = 21 \Rightarrow BC = \frac{21}{5}.}$$

3. In *Figure 3*, $\sin\angle EDF = \frac{3}{5}$ and $\sin\angle DEF = \frac{4}{5}$.

If $DF = 7$, find the length of $\overline{EF}$.

Figure 3



$\boxed{\frac{21}{4}}$, **or** $\boxed{5.25}$.

Notice that $\triangle DFT$ is identical to $\triangle ABC$ from problem #2! So the altitude $\overline{FT}$ has length $\frac{21}{5}$, and: $\sin\angle DEF = \frac{4}{5} = \frac{21/5}{EF} \Rightarrow 4 \cdot EF = 21 \Rightarrow EF = \frac{21}{4}$.

4. Again in *Figure 3*, $\sin\angle EDF = \frac{3}{5}$ and $\sin\angle DEF = \frac{4}{5}$, but DF does not necessarily equal 7.

If $DE = 10$, find the length of the altitude dropped from F.

$\boxed{\frac{24}{5}}$, **or** $\boxed{4.8}$.

Let $FT = 3x$, and $DF = 5x$. Then $\sin\angle DEF = \frac{4}{5} = \frac{3x}{EF} \Rightarrow EF = \frac{15}{4}x$.

Note that $FT = 3x = \frac{12}{4}x$. *This means that $\triangle FET$ has sides in a 3:4:5 ratio, and so* $TE = \frac{9}{4}x$. *Similar reasoning indicates that $DT = 4x$. Since $DT + TE = DE$,*

$$4x + \frac{9}{4}x = 10 \Rightarrow \frac{25}{4}x = 10 \Rightarrow x = \frac{40}{25} = \frac{8}{5}, \text{ so } FT = 3x = 3\left(\frac{8}{5}\right) = \frac{24}{5}.$$

Minnesota State High School Mathematics League

Individual Event

2009-10 Event 1D
SOLUTIONS

1. In the following synthetic division, what number belongs in the empty box?

$\boxed{4}$.

$$\begin{array}{cccccc} \underline{-2} & 1 & 7 & \boxed{} & -9 \\ & 1 & 5 & -6 & \boxed{3} \end{array}$$

Concentrate on the circled values. They lead us to the equation: $(-2)(5)+x=-6$. *Solving, x = 4.*

2. Find the remainder when $2x^3-x+6$ is divided by $x-5$.

$\boxed{251}$.

Using the Remainder Theorem, we can substitute 5 for x: $2(5)^3-(5)+6=250+1=251$.

3. If $f(x)=x+1$, $g(x)=x^2+2$, and $h(x)=x^3+3$, write an expression for $f\left(g\left(h(x)\right)\right)$ as a standard-form polynomial in *x*.

$\boxed{x^6+6x^3+12}$.

Work from the inside out: $g\left[h(x)\right]=g\left[x^3+3\right]=\left[x^3+3\right]^2+2=x^6+6x^3+11$,

So $f\left(g\left(h(x)\right)\right)=f\left(x^6+6x^3+11\right)=\left(x^6+6x^3+11\right)+1=x^6+6x^3+12$.

4. For how many integers $c\le 2009$ will the solutions of the equation $x^2-38x+c=0$ be complex conjugates of the form $a\pm bi$, where *a* and *b* are positive integers?

$\boxed{40}$.

The sum of the roots = $(a+bi)+(a-bi)=2a=38 \Rightarrow a=19$.

The product of the roots = $(a+bi)(a-bi)=a^2+b^2=19^2+b^2=c\le 2009$.

So $361+b^2\le 2009$*, and* $b^2\le 1648$. *Since b is a positive integer, we are essentially being asked for the number of perfect squares less than or equal to 1648.*

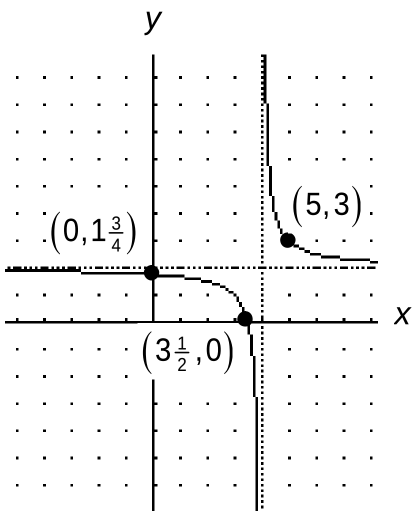
$\sqrt{1648}\approx 40.6$*, so the number of integral values of c is 40 .*

Minnesota State High School Mathematics League

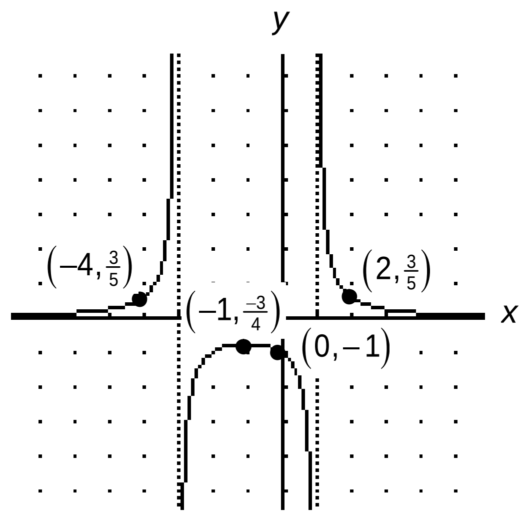
Team Event

2009-10 Meet 1
SOLUTIONS

1.



$$f(x)=\frac{a}{x-c}+d$$



$$g(x)=\frac{p}{(x-q)(x-r)}$$

The graphs of *f(x)* and *g(x)*, two rational functions, are shown. The function *(f+g)(x)* has exactly one rational root. Find it.

$\boxed{\frac{1}{2}}$ ***, or*** $\boxed{0.5}$.

2. The greatest common divisor of *a*, *b*, and *c* is 6. The greatest common divisor of *ab*, *ac*, and *bc* is 360. What is the smallest possible value for *abc*?

$\boxed{21600}$.

3. For a circle of radius *r*, the ratio of the side length of a regular circumscribed polygon of *n* sides to the side length of a regular inscribed polygon of *n* sides may be written in the form $T(f(n))$, where *T* is one of the six standard trigonometric functions and *f* is a function involving π and *n.* Do so.

$\boxed{\sec\left(\frac{\pi}{n}\right)}$.

4. If $f(x)=x+1$, $g(x)=x^2+2$, $h(x)=x^3+3$, and $k(x)=x-c$, find the <u>smallest</u> value of c that causes $k\left(f\left(g\left(h(x)\right)\right)\right)$ to have a real root.

$\boxed{3}$.

5. In parallelogram *PQRS*, angle *PSR* is acute, and point *X* is located on side $\overline{RS}$ so that *XR* = *RQ*, *XQ* = *QP*, and *XP* = *PS*. Find $m\angle PSX$.

$\boxed{72^\circ}$.

6. Express $\dfrac{5}{0.\overline{2}+0.\overline{4}+0.\overline{7}}$ as the quotient of two relatively prime integers.

$\boxed{\dfrac{45}{13}}$.

Graders' note:
<u>No</u> alternate forms accepted here!

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1. *Using the asymptotes,* $f(x)=\dfrac{a}{x-4}+2$ *and* $g(x)=\dfrac{p}{(x-1)(x--3)}$.

Using the y-intercepts, $\dfrac{7}{4}=\dfrac{a}{-4}+2 \Rightarrow a=1$ *and* $-1=\dfrac{p}{(-1)(3)} \Rightarrow p=3$.

$(f+g)(x)=\dfrac{1}{x-4}+2+\dfrac{3}{(x-1)(x+3)}$. *Simplifying the right side yields* $\dfrac{2x^3-3x^2-17x+9}{(x-4)(x-1)(x+3)}$.

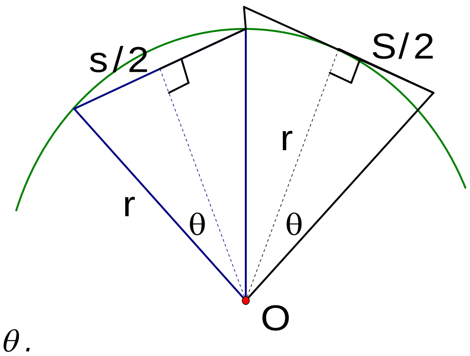
The numerator controls the roots, so we consider the equation $2x^3-3x^2-17x+9=0$.
The possible rational roots are $\pm 1,3,9,\frac{1}{2},\frac{3}{2},\frac{9}{2}$. *Graphing* $y=2x^3-3x^2-17x+9$, *there are real roots between 3 and 4, 0 and 1, and –3 and –2. Only one of these intervals contains a rational root possibility; the interval 0 to 1 contains* $\frac{1}{2}$. *Test:*

$\frac{1}{2}$	2	–3	–17	9
	2	–2	–18	$\boxed{0}$

rational root = $\boxed{\dfrac{1}{2}}$

2. *a, b, and c must all be multiples of both 2 and 3, so each of* $\{ab,ac,bc\}$ *is a multiple of* $2^2\cdot 3^2$.
Since $360=2^3\cdot 3^2\cdot 5$, *we need to figure out where the 5 and the extra power of 2 come from.*
We can deduce that exactly two of $\{a,b,c\}$ *are multiples of 5. (If all three were, then the GCD would contain a factor of 5; if only one was a factor of 5, then the product of the other two could not be a multiple of 360.) Using similar reasoning, exactly two of* $\{a,b,c\}$ *are multiples of 4. Therefore, without loss of generality, we know the following:*
a and b both contain factors of 3, 4, and 5; c contains factors of 2 and 3.
So the smallest possible value for abc is $(3\cdot4\cdot5)\cdot(3\cdot4\cdot5)\cdot(2\cdot3)=\boxed{21600}$. *(a = b = 60; c = 6)*

3. *Consider the figure shown at right, which displays one of the congruent triangle "sections" from the inscribed polygon, adjacent to a similar section from the circumscribed polygon. In each case, the central angle enclosed by each section is* $\frac{2\pi}{n}$*, so* $\theta = \frac{\pi}{n}$.



In the inscribed section, $\sin\theta = \frac{s/2}{r} = \frac{s}{2r} \Rightarrow s = 2r\sin\theta$.

In the circumscribed section, $\tan\theta = \frac{S/2}{r} = \frac{S}{2r} \Rightarrow S = 2r\tan\theta$.

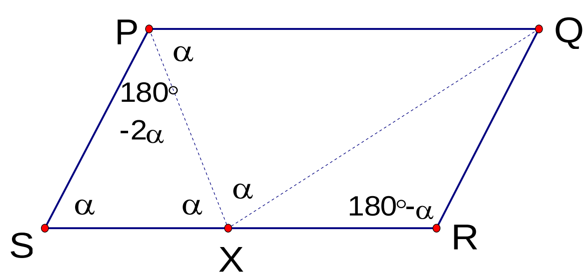
$$\frac{S}{s} = \frac{\cancel{2r}\tan\theta}{\cancel{2r}\sin\theta} = \frac{1}{\cos\theta} = \sec\theta = \boxed{\sec\left(\frac{\pi}{n}\right)}.$$

4. *From the solution to problem 3 of Event D,* $f\left(g\left(h(x)\right)\right) = x^6 + 6x^3 + 12$.

So $k\left(f\left(g\left(h(x)\right)\right)\right) = x^6 + 6x^3 + (12 - c)$. *Letting* $y = x^3$*, this converts to* $y^2 + 6y + (12 - c)$.

Applying the Quadratic Formula, $y = \frac{-6 \pm \sqrt{36 - 4(12 - c)}}{2}$. *To have a real root, the discriminant must be non-negative:* $36 - 4(12 - c) = 4c - 12 \ge 0 \Rightarrow c \ge \boxed{3}$.

5. *Label the desired angle as* α. *Then label the other angles of* $\triangle PSX$ *as shown. Since* $\angle PSX$ *is supplementary with both* $\angle XRQ$ *and* $\angle SPQ$, $m\angle XRQ = m\angle SPQ = 180^\circ - \alpha$. *This means* $m\angle XPQ$ *also equals* α*!* $(\triangle SPX \sim \triangle PQX)$



Now, $m\angle PQR = m\angle PQX + m\angle XQR$

$$\alpha = (180^\circ - 2\alpha) + \frac{\alpha}{2}$$

$$\frac{5\alpha}{2} = 180^\circ \Rightarrow \alpha = \boxed{72^\circ}.$$

6. $$\frac{5}{0.\overline{2} + 0.\overline{4} + 0.\overline{7}} = \frac{5}{\left(\frac{2}{9} + \frac{4}{9} + \frac{7}{9}\right)} = \frac{5}{\left(\frac{13}{9}\right)} = 5 \cdot \frac{9}{13} = \boxed{\frac{45}{13}}.$$