Minnesota State High School Mathematics League

Individual Event

2008-09 Event 2A
SOLUTIONS

1. Given that $(y+7)+(2y+x^2)=(3+x^2)+3$, find the value of y.

y* =** $\boxed{\dfrac{-1}{3}}$ ***or $\boxed{-0.333}$.

$3y+7+x^2=6+x^2 \;\Rightarrow\; 3y=-1$, *etc.*

2. The Sir Charge car rental company charges \$45 per day to rent a car, plus \$6.00 per gallon of gas used. The car can drive 26 miles per gallon. If Teddy rents this car from Sir Charge for 5 days and drives it a total of *m* miles, write a linear expression in terms of *m* which describes Teddy's total cost.

$\boxed{225+\dfrac{3}{13}m}$ **.**

$\dfrac{\$45}{\cancel{day}}\cdot 5\ \cancel{days}=\$225$; $\quad m\ \cancel{miles}\cdot\dfrac{1\ \cancel{gallon}}{26\ \cancel{miles}}\cdot\dfrac{\$6}{\cancel{gallon}}=\dfrac{3}{13}m$.

3. Don's calculator has some warped circuits and does not input integers correctly. When Don enters an odd integer *n*, the calculator interprets it as *n* + 3. When Don enters an even integer *m*, the calculator interprets it as *m*/2 . If Don entered the numbers 15, 24, 33, and *x*, and the calculator claimed the sum of those numbers was 101, find *x*.

***x* =** $\boxed{70}$ **.**

If x is odd, then the sum is $(15+3)+(24/2)+(33+3)+(x+3)=101$.
Solving, x = 32, which contradicts the fact that x is odd.
If x is even, then the sum is $(15+3)+(24/2)+(33+3)+(x/2)=101$.
Solving here yields x = 70.

4. Ole, who occasionally lies, wrote down an integer *N*, and then gave his friend Lena the following inequalities:

A. $N-13<50-N$ **B.** $N+8<100-2N$
C. $3N+60<5N+3$ **D.** $5N-20<N+99$

As it turned out, exactly two of those statements were lies. Find the value of *N*.

***N* =** $\boxed{31}$ **.**

Solving, ***A.*** $N<31.5$, ***B.*** $N<30\frac{2}{3}$, ***C.*** $N>28.5$, ***D.*** $N<29.75$.
A *is true, since otherwise it would make both* ***B*** *and* ***D*** *false.*
C *is true, since no other statements can be false with it. So* $30\frac{2}{3}<N<31.5$.

Minnesota State High School Mathematics League

Individual Event

2008-09 Event 2B
SOLUTIONS

1. The lengths of the sides of a scalene triangle, listed in size order, are 5, x, and 15. How many possible values for x are there, given that x is an integer?

There are $\boxed{4}$ ***possible values.***

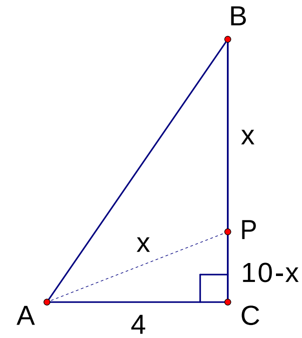
By the Triangle Inequality, $5+x>15 \Rightarrow x>10$.
Since the sides are in size order, x = 11, 12, 13, or 14.

2. Point P is chosen along leg $\overline{BC}$ of right triangle ABC so that $BP = PA$ *(Figure 2).* If $BC = 10$ and $AC = 4$, find BP.

BP = $\boxed{\frac{29}{5}}$ ***or*** $\boxed{5\frac{4}{5}}$ ***or*** $\boxed{5.8}$ **.**

Labeling as shown in the diagram,

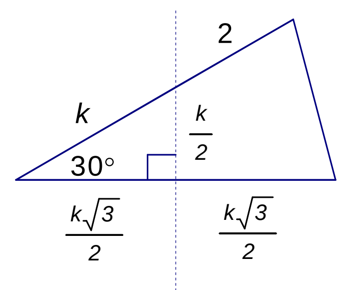
$$4^2+(10-x)^2=x^2$$
$$16+\left(100-20x+x^2\right)=x^2$$
$$116=20x$$



3. In an isosceles triangle with a 30° vertex angle, the perpendicular bisector of one leg divides the other leg into the ratio $k : 2$. If $k \geq 2$, find k.

k = $\boxed{\frac{2}{\sqrt{3}-1}}$ ***or*** $\boxed{\sqrt{3}+1}$ ***or*** $\boxed{2.732}$ **.**

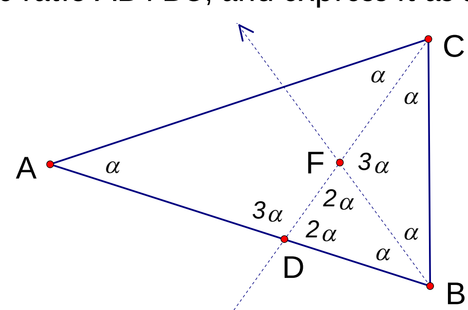
$$k+2=\frac{k\sqrt{3}}{2}+\frac{k\sqrt{3}}{2}$$
$$k+2=k\sqrt{3}$$
$$2=k\left(\sqrt{3}-1\right)$$



4. In triangle ABC, $m\angle ABC = m\angle ACB = 72°$. Find the ratio $AB : BC$, and express it as a decimal accurate to three decimal places.

AB : BC = $\boxed{1.618}$ ***or*** $\boxed{\left(1+\sqrt{5}\right):2}$ ***or*** $\boxed{2:\left(\sqrt{5}-1\right)}$

Labeling angles, $\alpha = 36°$, $2\alpha = 72°$, *etc.*
Also, let BF = FC = BD = x, and DF = y. Set $\frac{AB}{BC}=r$.
$\Delta CDB \sim \Delta ABC \Rightarrow \frac{CD}{DB}=\frac{x+y}{x}=1+\frac{1}{r}=\frac{AB}{BC}=r$.
Use a calculator's equation solver on $1+\frac{1}{r}=r$.



Minnesota State High School Mathematics League

Individual Event

2008-09 Event 2C

SOLUTIONS

Questions 1 and 2 refer to Figure 1 in which $\tan\alpha = \frac{5}{12}$, $\tan\beta = 4\sqrt{5}$.

1. Find $\sin 2\alpha$.

$\boxed{\frac{120}{169}}$ ***or*** $\boxed{.710}$.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right)$$

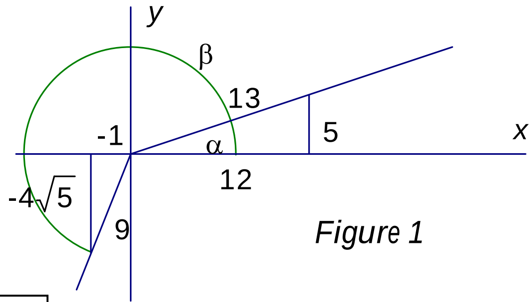


Figure 1

2. Find $\cos\frac{\beta}{2}$.

$\boxed{\frac{-2}{3}}$ ***or*** $\boxed{-.667}$.

$$\cos\frac{\beta}{2} = \pm\sqrt{\frac{1+\cos\beta}{2}} = \pm\sqrt{\frac{1-\frac{1}{9}}{2}} = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$$

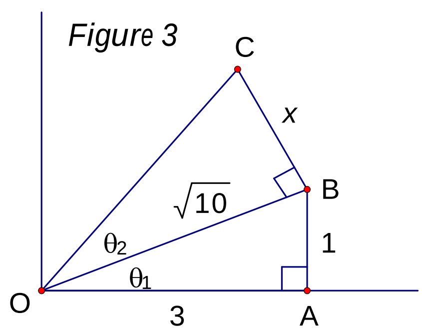


Figure 3

3. In Figure 3, $\theta_1 + \theta_2 = 45°$. Find BC.

$\boxed{\frac{\sqrt{10}}{2}}$ ***or*** $\boxed{1.581}$.

$$\tan\theta_2 = \tan\left(45° - \theta_1\right) = \frac{\tan 45° - \tan\theta_1}{1 + (\tan 45°)(\tan\theta_1)} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2}$$

$$x = \sqrt{10}\cdot\tan\theta_2 = \sqrt{10}\cdot\frac{1}{2}$$

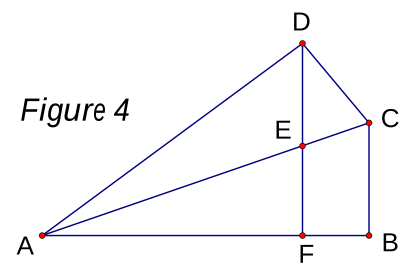


Figure 4

4. *ABCD* is a kite-shaped quadrilateral *(Figure 4)* with *BC* = *CD* and *AB* = *AD* = *x*. $m\angle BAD = 45°$; $m\angle BCD = 135°$. From *D*, drop a line perpendicular to $\overline{AB}$, meeting $\overline{AC}$ at *E*, and $\overline{AB}$ at *F*. Find *EF* in terms of *x*.

EF **=** $\boxed{x\cdot\frac{2-\sqrt{2}}{2}}$ ***or*** $\boxed{0.293\cdot x}$

$$\tan\frac{45°}{2} = \frac{BC}{x} = \frac{1-\cos 45°}{\sin 45°} = \frac{1-(\sqrt{2}/2)}{(\sqrt{2}/2)} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1$$

$$\sin 45° = \frac{FD}{x} \Rightarrow FD = x\cdot\frac{\sqrt{2}}{2} = AF$$

$$\frac{AF}{AB} = \frac{EF}{BC} \Rightarrow \frac{x\cdot\frac{\sqrt{2}}{2}}{x} = \frac{EF}{x(\sqrt{2}-1)} \Rightarrow EF = x\cdot\left(\sqrt{2}-1\right)\cdot\frac{\sqrt{2}}{2}, \textit{ etc.}$$

Minnesota State High School Mathematics League

Individual Event

2008-09 Event 2D

SOLUTIONS

1. Write the equation of the line which passes through the origin and is parallel to $2x-3y=7$.

$\boxed{y=\frac{2}{3}x}$ **or** $\boxed{2x-3y=0}$.

$2x-3y=7$ has slope $\frac{-A}{B}=\frac{-(2)}{(-3)}=\frac{2}{3}$,
and any line through the origin has y-intercept = 0.

2. Find the intersection point of the lines $y=\frac{7}{5}x-10$ and $y=\frac{3}{4}x$.

(x , y) = $\boxed{\left(\frac{200}{13},\frac{150}{13}\right)}$ **or** $\boxed{(15.385,\ 11.538)}$

$$\frac{3}{4}x=\frac{7}{5}x-10 \Rightarrow \left(\frac{15}{20}-\frac{28}{20}\right)x=-10$$

$$\frac{-13}{20}x=-10 \Rightarrow x=\frac{200}{13}$$

$$y=\frac{3}{4}x=\frac{3}{\cancel{4}}\cdot\frac{\overset{50}{\cancel{200}}}{13}=\frac{150}{13}$$

3. Find all ordered pairs (x,y) that satisfy

$$\begin{cases} x^2+y^2=5 \\ x^2+(y+4)^2=6^2 \end{cases}$$

(x , y) = $\boxed{\left(\pm\frac{\sqrt{95}}{8},\frac{15}{8}\right)}$

Subtracting eqn. 1 from eqn. 2 yields

$$y^2+8y+16-y^2=36-5$$

$$8y=15 \Rightarrow y=\frac{15}{8}$$

$$x^2+\left(\frac{15}{8}\right)^2=x^2+\frac{225}{64}=5 \Rightarrow x^2=\frac{95}{64}$$

<u>Graders:</u> deduct 1 point for omitting ±, <u>or</u> for placing ± on the y-coordinate.
(If a student makes both mistakes, give no credit.)

4. Two lines intersect in the *xy*-plane. The first line has *x*-intercept 2*p* and *y*-intercept 2*p*, while the second line has *x*-intercept *p* and *y*-intercept 3*p*. If their intersection point is concurrent with the line *x* = 3, find *p*. *(Credit: 1998 NC HS Math Contest)*

p = $\boxed{6}$.

$$\begin{cases} \frac{x}{2p}+\frac{y}{2p}=1 \\ \frac{x}{p}+\frac{y}{3p}=1 \end{cases} \Rightarrow \begin{cases} x+y=2p \\ 3x+y=3p \end{cases} \Rightarrow 2x=p=6\text{, since } x=3.$$

Minnesota State High School Mathematics League

Team Event

2008-09 Meet 2
SOLUTIONS

1. If a, b, and c are distinct in the system $\begin{cases} a^3 + 3a = -14 \\ b^3 + 3b = -14 \\ c^3 + 3c = -14 \end{cases}$, find the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

$\boxed{\frac{-3}{14}}$ ***or*** $\boxed{-0.214}$.

2. The altitude and the median from vertex A of triangle ABC are 4 and 5 units long, respectively. The altitude bisects the angle determined by side $\overline{AB}$ and the median. Find AC.

$\boxed{\sqrt{97}}$ ***or*** $\boxed{9.849}$.

3. Referring to problem #3 from Event A, Don bought a new calculator to replace his old one. Sadly, this new calculator also inputs integers incorrectly. It interprets the odd integer n as $4n$, and the even integer m as $m + 5$. When Don used the new calculator to multiply the positive integers a and b, the result was 4444. Find the smallest possible value that the <u>calculator</u> would return for the sum $a + b$.

$\boxed{145}$.

4. Susan, who had no calculator available, was asked to solve $\sqrt{6}\cos x + \sqrt{2}\sin x = 2$. She squared both sides and simplified to get the equation $2\cos^2 x + b\sin 2x = 1$. In exact form, what is b ?

$\boxed{2\sqrt{3}}$. ***<u>Graders:</u> Note that the problem requires this answer in <u>exact form</u>.***

5. Points A and B are on the same side of line m and are 5 and 7 units away from m, respectively. A and B are 17 units apart. For all points P on line m, what is the smallest possible value of $AP + BP$?

$\boxed{\sqrt{429}}$ ***or*** $\boxed{20.712}$.

6. In a triangle, the lengths of the three medians are 9, 12, and 15. Find the length of the side to which the longest median is drawn.

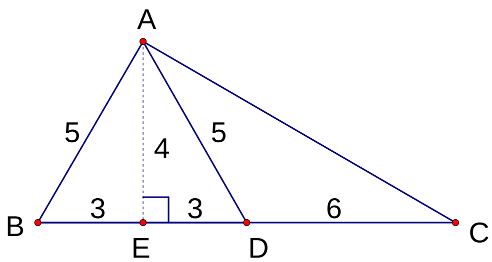
$\boxed{10}$.

1. *a, b, and c are actually the three roots of* $x^2+3x+14=0$.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+ac+bc}{abc}=\boxed{\frac{-3}{14}}.$$

(This last step uses Vieta's Formulas, which relate the coefficients of a polynomial to its roots. For more info:)

http://www.artofproblemsolving.com/Resources/Papers/PolynomialsAK.pdf

2. 

Drawing a good picture can make problems easy: Triangles AED and AEB are both 3-4-5, and since $\overline{AD}$ *is a median, BD = CD = 6. Applying the Pyth. Thm. to right triangle AEC,*

$$x^2=4^2+9^2=97 \quad\Rightarrow\quad x=\boxed{\sqrt{97}}.$$

3. *Don's calculator turns even numbers into odds, and odds into multiples of 4. So, for a product to equal* $4444=4\cdot1111$, *one of the original numbers must be odd (to become a multiple of 4), while the other must be even (to become odd by adding 5). Suppose a is odd and b is even. Then* $4a(b+5)=4444 \Rightarrow a(b+5)=1111$. *Either* $1111=11\cdot101$, *or* $1111=1\cdot1111$. *Since b is positive,* $b+5$ *can only equal 11, 101, or 1111, yielding b = 6, 96, or 1106. This forces a = 101, 11, or 1, respectively. Don's calculator will output a + b as (4a) + (b + 5). Checking all three possible pairs of integers (a, b), the smallest sum is* $(4\cdot11)+(96+5)=\boxed{145}$.

4.

$$\left(\sqrt{6}\cos x+\sqrt{2}\sin x\right)^2=(2)^2$$

$$6\cos^2 x+2\sqrt{12}\sin x\cos x+2\sin^2 x=4$$

$$4\cos^2 x+\left(2\cos^2 x+2\sin^2 x\right)+2\sqrt{3}\left(2\sin x\cos x\right)=4$$

$$4\cos^2 x+\left(2\right)+2\sqrt{3}\sin 2x=4$$

$$4\cos^2 x+2\sqrt{3}\sin 2x=2$$

$b=\boxed{2\sqrt{3}}$.

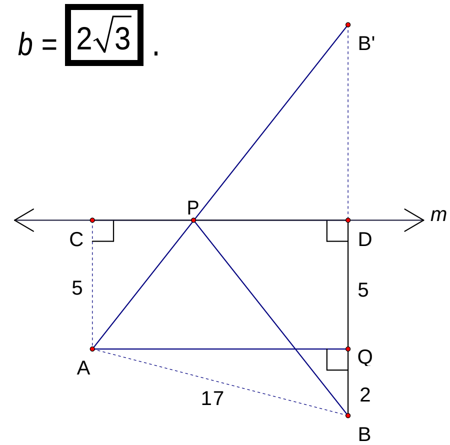
5.

$$AP+PB=AP+PB'=\sqrt{B'Q^2+QA^2}$$

$$=\sqrt{12^2+\left(AB^2-BQ^2\right)}$$

$$=\sqrt{144+\left(17^2-2^2\right)}$$

$$=\boxed{\sqrt{429}}$$



6. *In the diagram, median* $\overline{BE}$ *is extended to point P so that ME = EP. The lengths of AM, ME, and MC are all determined by the fact that the centroid divides each median into segments of length ratio 2 : 1. Since AC and MP bisect each other, AMCP is a parallelogram. Further, AMP is a 6-8-10 triangle, so* $\angle MAP=90^\circ$, *and AMCP is a rectangle! AC = MP =* $\boxed{10}$.