Minnesota State High School Mathematics League

Individual Event

2008-09 Event 1A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{5}{6}$ ______ 1. Express $\frac{1}{2}+\frac{1}{4}+\frac{1}{12}$ as the quotient of two relatively prime numbers.

$\frac{4}{21}$ ______ 2. Express 12.5% of $\frac{.0032}{.0018+.0003}$ as the quotient of two relatively prime numbers.

98% ______ 3. [Here is a slight modification of a problem credited to the well known mathematician, Paul Halmos] A watermelon weighs 500 pounds, 99% of its weight being due to the water it contains. After it sat in a drying room for a while, it lost 250 pounds of water. What percent of its weight was then water?

4. Three positive integers *L*, *M*, and *N* satisfying $L < M < N$, have a greatest common divisor of 12 and a least common multiple of 180. Find all possible triples (L,M,N).

(12, 36, 60) (12, 36, 180) (12, 60, 180), (36, 60, 180)

Graders: Award 1 point if only three, all correct, are given

1. $\frac{6+3+1}{12} = \frac{10}{12} = \frac{5}{6}$

2. $\frac{1}{8} \cdot \frac{32}{18+3} = \frac{4}{21}$

3. In the beginning, it contains $.99(500) = 495$ pounds of water. Later it contains $495 - 250 = 245$ pounds of water, and it weighs non-water content + water $= 5 + 245 = 250$ pounds.

% water $= \frac{245}{250} = .98$

4. g.c.d $= 2\cdot2\cdot3$

l.c.m $= 2\cdot2\cdot3\cdot3\cdot5$

Possibilities:

L	M	N
$2\cdot2\cdot3$	$2\cdot2\cdot3\cdot3$	$2\cdot2\cdot3\cdot5$
$2\cdot2\cdot3$	$2\cdot2\cdot3\cdot3$	$2\cdot2\cdot3\cdot3\cdot5$
$2\cdot2\cdot3$	$2\cdot2\cdot3\cdot5$	$2\cdot2\cdot3\cdot3\cdot5$
$2\cdot2\cdot3\cdot3$	$2\cdot2\cdot3\cdot5$	$2\cdot2\cdot3\cdot3\cdot5$

Triples are, therefore

(12, 36, 60)

(12, 36, 180)

(12, 60, 180)

(36, 60, 180)

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Individual Event

2008-09 Event 1B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

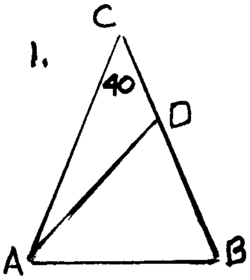
$30°$ 1. The isosceles ΔABC in Figure 1 has vertex $\angle C = 40°$. A point D is chosen on BC so that $AD = AB$. What is the measure in degrees of $\angle DAC$?

$135°$ 2. In the right ΔABC, the bisectors of the acute angles B and C meet at D to form an isosceles ΔBCD. What is the measure in degrees of $\angle BDC$?

$36°$ 3. The vertices of a regular pentagon, labeled in a counterclockwise direction, are *ABCDE*. What is the angle measure of $\angle DAE$?

$21°$ 4. In quadrilateral ABCD (Figure 4), $\angle ABC = 42°$. Furthermore, if AB is extended to E so that $AB = BE$, then $\angle ACE = 90°$. What is the measure of $\angle AEC$?

1.



$\angle A = \angle B = \frac{180-40}{2} = 70$

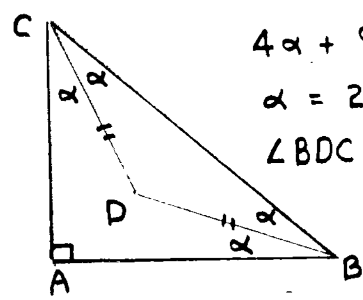
ΔABD is isosceles, so

$\angle BDA = 70°$, making

$\angle ADC = 110°$

$\angle DAC = 180 - (40 + 110) = 30$

2.

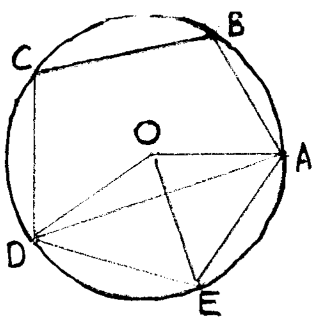


$4\alpha + 90 = 180$

$\alpha = 22\frac{1}{2}°$

$\angle BDC = 180 - 2\alpha = 135°$

3.



Let the center be O; $\angle AOD = 2\,\frac{360}{5} = 144$

so $\angle DAO = \frac{180-144}{2} = 18°$. $\angle OAE = \frac{180-72}{2} = 54$

$\angle DAE = \angle OAE - \angle DAO = 54 - 18 = 36$

4.

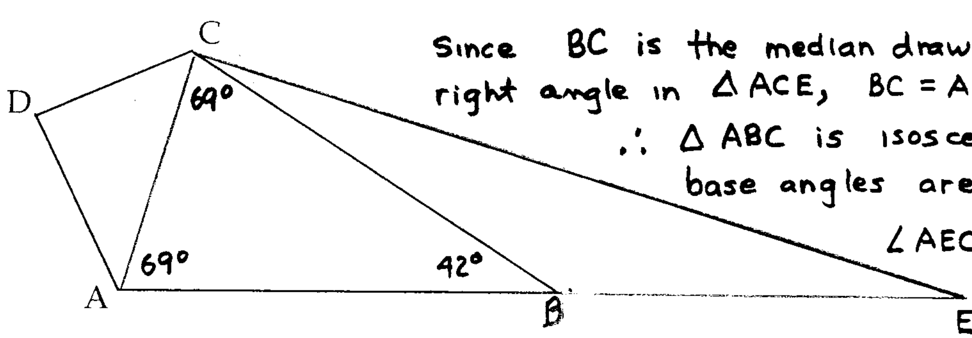


Figure 4

Since BC is the median drawn from the right angle in ΔACE, $BC = AB = BE$

$\therefore$ ΔABC is isosceles, and its base angles are $\frac{180-42}{2} = 69°$

$\angle AEC + 69° = 90$

$\angle AEC = 21°$

Minnesota State High School Mathematics League

Individual Event

2008-09 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$-\frac{3}{20}$ 1. For the second quadrant angle pictured in Figure 1, find $\sin\alpha + \tan\alpha$.

$\frac{7\pi}{6}$ 2. Express as a multiple of π the radian measure of an angle in the third quadrant that has a sine of $-\frac{1}{2}$.

3. Figure 3 shows the graph, but without scales on the axes, of $y = 2\sin\frac{4}{3}x$. After placing scales on the axes, give the letter labeling the point on the graph having an x-coordinate of

(a) π E (b) $\frac{\pi}{2}$ C

2850 4. Round to the nearest multiple of 50 the number of x intercepts on the graph of $y = \sin\frac{1}{x}$ when $0.0001 < x < 0.001$. That is, to the nearest 50, how many times will the graph of $y = \sin\frac{1}{x}$ cross the x-axis between 0.0001 and 0.001?

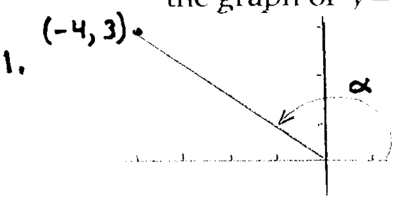
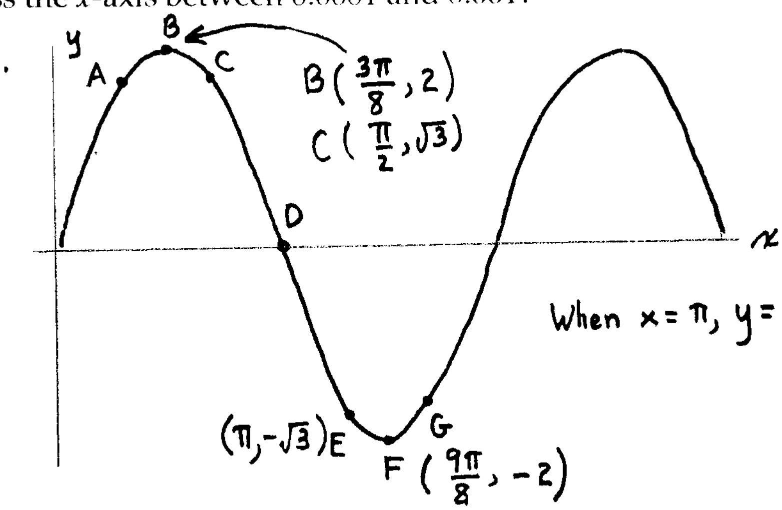
1. 

Figure 1

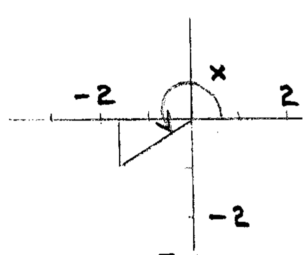
$\sin\alpha + \tan\alpha = \frac{3}{5} - \frac{3}{4}$

$= \frac{12-15}{20} = -\frac{3}{20}$

3. 

When $x = \pi$, $y = 2\sin\frac{4\pi}{3}$

2. $\sin x = -\frac{1}{2}$



$x = \frac{7\pi}{6}$

4. We seek k for which $\frac{1}{10{,}000} < \frac{1}{k\pi} < \frac{1}{1000}$

$k\pi < 10{,}000 \Rightarrow k < \frac{10{,}000}{\pi} \approx 3183$

$k\pi > 1{,}000 \Rightarrow k > \frac{1000}{\pi} \approx 318$

$2865 \approx 2850$

Minnesota State High School Mathematics League

Individual Event

2008-09 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$1, -5$ 1. Find all the solutions to $x^2 + 4x + 4 = 9$.

$-\frac{9}{2}, 5$ 2. Find all the solutions to $(2x-3)(x+1) = 42$.

3. Write the equation of the parabola passing through (3,7), (1,4) and (5,4). Write your answer in the form $y = ax^2 + bx + c$ OR $x = ay^2 + by + c$, whichever form fits the situation.

$y = -\frac{3}{4}x^2 + \frac{9}{2}x + \frac{1}{4}$

$-\frac{4}{3}$ 4. Find the smallest root of $6x^3 - 13x^2 - 19x + 12 = 0$

1. $(x+2)^2 = 9$

$x + 2 = \pm 3$

$x = 1, -5$

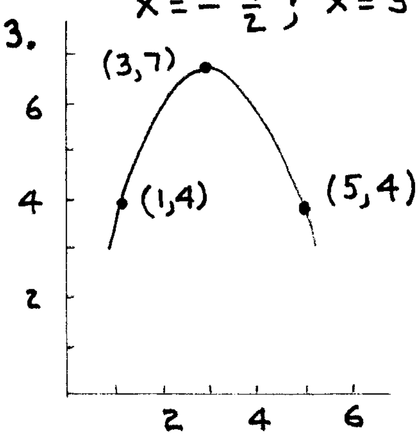
2. $2x^2 - x - 3 = 42$

$2x^2 - x - 45 = 0$

$(2x+9)(x-5) = 0$

$x = -\frac{9}{2}$; $x = 5$

3.



$y - 7 = a(x-3)^2$

When $x=1$, $y = 4$

$4 - 7 = a(4)$

$a = \frac{-3}{4}$

$4(y-7) = -3(x^2 - 6x + 9)$

$4y - 28 = -3x^2 + 18x - 27$

$y = -\frac{3}{4}x^2 + \frac{9}{2}x + \frac{1}{4}$

4.

	6	-13	-19	12	
3	6	5	-4		0

$6x^2 + 5x - 4 = 0$

$(3x+4)(2x-1) = 0$

$x = \frac{-4}{3}$, $x = \frac{1}{2}$

Minnesota State High School Mathematics League

Team Event

2008-09 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

55° ______ 1. Isosceles ΔABC (Figure 1) has base angles $\angle A = \angle B = 70°$. *AE* makes an angle of θ with the *AB*, and θ varies as *E* moves up and down *BC*. *DE* is parallel to *AB*, and of course it too moves up or down with *E*. The extensions of *DE* to *DF* and *AE* to *AG* form angles $\alpha = \angle FEG$ and $\beta = \angle GEC$. What will be the measure of θ when $\alpha = \beta$? {AHSME, 1968, Number 18]

\$500 ______ Lost OR ______ Gained 2. Having purchased 200 shares of a stock at one price, and another 200 shares at a higher price, Mr. Gotbucks later sold all 400 shares for \$30 each. He thereby gained 20% on the first 200 shares, but lost 20% on the other 200 shares. How much did he gain or lose at the time of the sale?

6 ______ 3. How many ordered pairs (*a*,*b*) of positive integers exist such that $\frac{1}{a}+\frac{5}{b}=\frac{1}{2}$?

360° ______ 4. Find the measure in degrees of the sum of angles *A*, *B*, *C*, *D*, E and *F* in Figure 4. [AHSME, 1972, Number 21]

663 ______ 5. Consider the set of composite positive integers between 47 and the next largest prime. Let *L* be the least common multiple of this set, and let *S* be the largest integer such that S^2 is a factor of L. What is the value of $\frac{L}{S^2}$?

10° ______ 6. In ΔADE, $\angle ADE = 140°$, points B and C lie on sides AD and AE respectively, and point A, B, C, D, and E, are distinct. If $AB = BC = CD = DE$, what is the measure of $\angle EAD$? [AHSME, 1978, Number 12]

Team ______________________________

Team Event 1 Solutions

1.

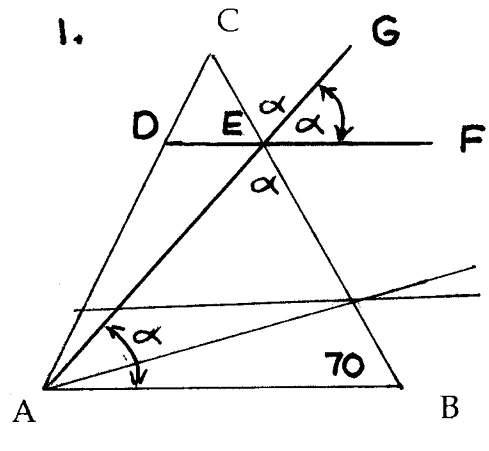


Figure 1

When $\alpha = \angle FEG = \angle GEC$

then $\angle AEB = \alpha$ (vertical $\angle$'s)

and $\angle BAE = \angle FEG = \alpha$ (corresp. $\angle$'s)

so $2\alpha + 70 = 180 \Longrightarrow \alpha = 55°$

4. Let $\alpha = \angle FPQ$, $\theta = \angle EQP$

From EFPQ,

(1) $F + \alpha + \theta + E = 360$

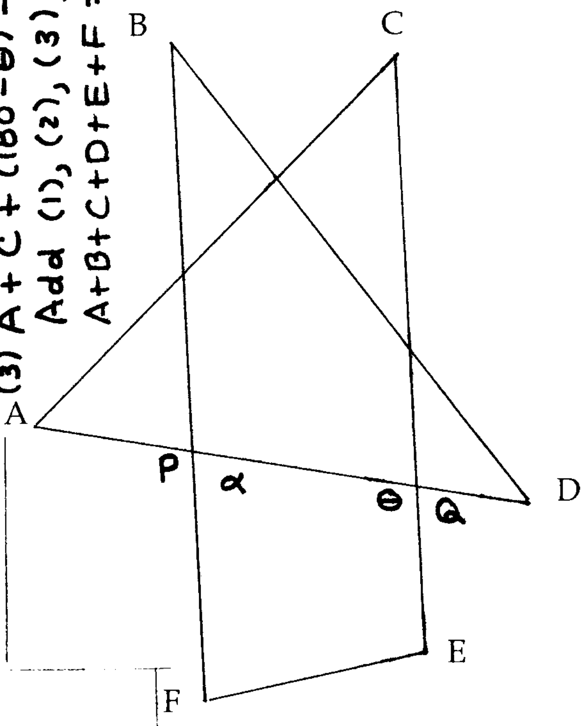
From ΔBCP and ΔACQ

(2) $B + (180 - \alpha) + D = 180$

(3) $A + C + (180 - \theta) = 180$

Add (1), (2), (3);

$A+B+C+D+E+F = 360$



2. The first 200 purchased at #x/share sold at #30/share. Gain $= 200(30-x) = .2(200x)$ Solving, purchase price $x = $ #25. He made $5(200) = $ #1000 on these shares.

The second 200 were purchased at #y/share, sold at #30/share. Loss $= 200(y-30) = .2(200y)$ Solving, purchase price $y = \frac{75}{2}$. He lost $\frac{15}{2}(200) = $ #1500. He lost #$(1500 - 1000) = $ #500

3. [Mass. Math Olympiad 2007-08]

$2b + 10a = ab$ so $a = \frac{2b}{b-10} = 2 + \frac{20}{b-10}$

Since a is an integer, $b - 10$ divides 20. See the table. Negative values for $b-10$ give negative values for either a or b.

$b-10$	b	$a = 2 + \frac{20}{b-10}$
1	11	22
2	12	12
4	14	7
5	15	6
10	20	4
20	30	3

5. The set under consideration is:

$48 = 2^4 \cdot 3$ $\quad 51 = 3 \cdot 17$

$49 = 7^2$ $\quad 52 = 2^2 \cdot 13$

$50 = 2 \cdot 5^2$

$LCM = 2^4 \cdot 3 \cdot 7^2 \cdot 5^2 \cdot 13 \cdot 17$

$= [\underbrace{2^2 \cdot 5 \cdot 7}_{S}]^2 \cdot 3 \cdot 13 \cdot 17$

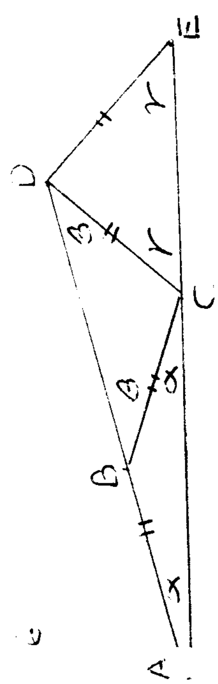
$\frac{L}{S^2} = 3 \cdot 13 \cdot 17 = 663$

Using the labels in the figure

$(180 - 2\gamma) + \beta = 140$

$(180 - 2\beta) + x + \gamma = 180$

$x + \gamma = 40$



Solve the system of three equations to get

$x = \angle TAD = 10$, $\beta = 20$, $\gamma = 30$