

# Minnesota State High School Mathematics League Individual Event

## 2008-09 Event 5A SOLUTIONS

1. The six-digit number  $\underline{2} \underline{1} \underline{7} \underline{X} \underline{8} \underline{5}$ , when divided by 9, leaves a remainder of 2. What is the value of the obscured digit,  $X$ ?

$X = \boxed{6}$ .

If  $\underline{2} \underline{1} \underline{7} \underline{X} \underline{8} \underline{5}$  leaves a remainder of 2 when divided by 9, then  $\underline{2} \underline{1} \underline{7} \underline{X} \underline{8} \underline{3}$  must be divisible by 9, and the sum of its digits must also be divisible by 9.  $2+1+7+X+8+3=21+X$ , which is only divisible by 9 if  $X=6$ .

2. A single digit is placed in each empty square in the grid (Figure 2) so that each row and each column contain exactly one of each of the digits 1, 2, 3, 4, and 5. What digit must be placed in the square at the bottom right corner?

	5	4		
1	3			
		5	3	
2		3	1	
5				3

Figure 2

$\boxed{3}$ .

In the left-most column, the only possible spot for the digit 5 is at the bottom. Now, in the bottom row, there is only one possible spot for the digit 3 ... the bottom right corner.

**Credit: Mathematics Teacher, Nov. 2006 Calendar Question #2.**

3. In the addition problem shown in Figure 3, each letter stands for a distinct non-zero digit. If the problem is mathematically correct, what is the largest possible value for the three-digit number **ONE**?

$$\begin{array}{r} \text{ONE} \\ + \text{ONE} \\ \hline \text{TWO} \end{array}$$

Figure 3

$\text{ONE} = \boxed{482}$ .

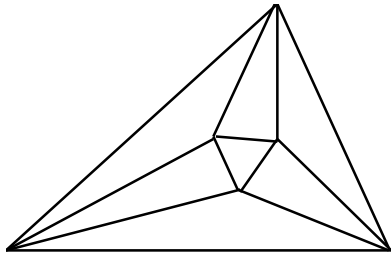
We concentrate on the hundreds digit (the "O") of **ONE**. The largest it can be is the digit 4. If  $O=4$ , then  $E=2$  or 7. We prefer 7, carrying a 1. Now make  $N$  as large as possible.  $N=9 \Rightarrow W=9$ ,  $N=8 \Rightarrow W=7$ , and  $N$  cannot be 7. Trying  $E=2$ ,  $N=9 \Rightarrow T=9$ , but  $N=8 \Rightarrow W=6, T=9$ . Yes!

4.  $N$  is a five-digit positive integer in which the first and last digits are unequal and nonzero.  $M$  is the five-digit integer obtained by writing the digits of  $N$  in reverse order. If  $M > N$ ,  $D = M - N$ , and the last three digits of  $D$  are  $\underline{9} \underline{7} \underline{6}$ , find  $D$ .

$D = \boxed{41976}$ .

Start by showing borrowing, with the idea that  $P > T$  implies  $T < P$ :

$$\begin{array}{r} PQRST \\ -TSRQP \\ \hline xy976 \end{array} \Rightarrow \begin{array}{r} P \overset{Q-1}{\cancel{0}} \overset{R+9}{\cancel{R}} \overset{S+9}{\cancel{S}} \overset{10+T}{\cancel{T}} \\ -T \quad S \quad R \quad Q \quad P \\ \hline x \quad y \quad 9 \quad 7 \quad 6 \end{array} \quad \begin{array}{l} (10+T)-P=6 \Rightarrow P-T=4. \\ (S+9)-Q=7 \Rightarrow Q-S=2 \Rightarrow y=1. \\ \text{So } Q \text{ needs no borrowing, and } x=4. \end{array}$$



# Minnesota State High School Mathematics League Individual Event

## 2008-09 Event 5B SOLUTIONS

1. The sides of square  $ABCD$  have length 3. Points  $R$ ,  $S$ ,  $T$ , and  $U$  partition the sides into lengths in the ratio 2:1, as shown in *Figure 1*. What is the area of  $\triangle ART$ ?

$$\text{Area}(\triangle ART) = \boxed{3}. \quad \text{Area}(\triangle ART) = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3.$$

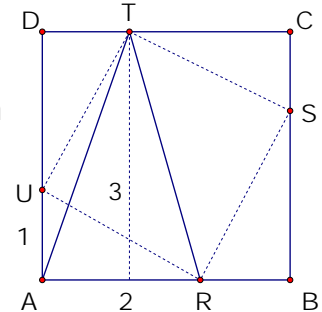


Figure 1

2. Again using *Figure 1* (as described in problem 1), find the area of  $RSTU$ .

$$\text{Area}(RSTU) = \boxed{5}. \quad \text{Area}(RSTU) = \text{Area}(ABCD) - 4 \cdot \text{Area}(\triangle ARU) = 3^2 - 4 \cdot \left(\frac{1}{2} \cdot 2 \cdot 1\right) = 5.$$

3. In *Figure 3*, each edge of square  $JKLM$  is partitioned into thirds. Four line segments are then drawn, connecting the partition points and forming an inner square  $WXYZ$ . Express, as a fraction in lowest terms, the portion of the area of  $JKLM$  covered by  $WXYZ$ .

$$\boxed{\frac{2}{9}}. \quad \text{Let } k = \text{the leg length of one of the small isosceles right triangles. Then } ZY = (2k\sqrt{2})\sqrt{2} - 2k = 4k - 2k = 2k, \text{ and } \frac{\text{Area}(WXYZ)}{\text{Area}(JKLM)} = \left(\frac{2k}{3k\sqrt{2}}\right)^2 = \frac{4}{18} = \frac{2}{9}.$$

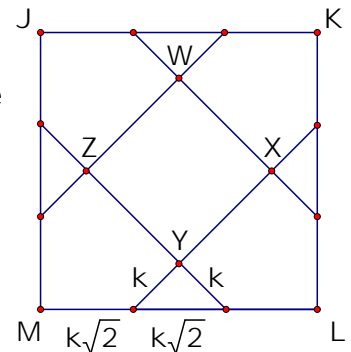
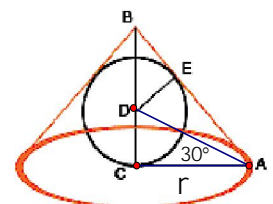
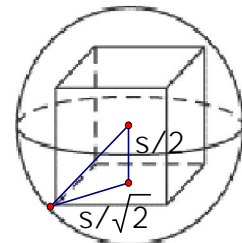
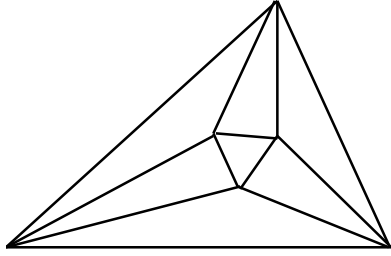


Figure 3

4. A cube of side length  $s$  is inscribed in a sphere, which is itself inscribed in a cone with slant height equal to the diameter of its base. Find the volume of the cone, expressed in terms of  $s$ .

$$\boxed{\frac{9\pi s^3 \sqrt{3}}{8}}. \quad r_{\text{sphere}} = \sqrt{(s/2)^2 + (s/\sqrt{2})^2} = \sqrt{(s^2/4) + (s^2/2)} = \frac{s\sqrt{3}}{2} = CD. \quad 30\text{-}60\text{-}90 \triangle ADC \Rightarrow r_{\text{cone}} = \sqrt{3} \cdot CD = \sqrt{3} \cdot \left(\frac{s\sqrt{3}}{2}\right) = 3s/2. \quad V_{\text{cone}} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \cdot r\sqrt{3} = \frac{\sqrt{3}}{3}\pi \left(\frac{3s}{2}\right)^3 = \frac{9\sqrt{3}}{8}\pi s^3.$$





# Minnesota State High School Mathematics League Individual Event

## 2008-09 Event 5C SOLUTIONS

1. How many distinct triangles exist within the MN State High School Math League logo at the top of this page?

**8**. *There are 3 obtuse triangles, 3 small "sliver" acute triangles, 1 equilateral triangle in the center, and... don't forget that the entire logo itself is 1 triangle!  $3+3+1+1 = 8$ .*

2. Find the number of positive-integer factors of 2600.

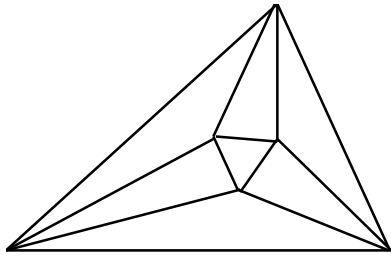
**24**.  $2600 = 2^3 \cdot 5^2 \cdot 13^1$ , which shows  $(3+1)(2+1)(1+1) = (4)(3)(2) = 24$  factors.

3. At a carnival game, a black velvet pouch contains either a gold or silver coin, with equal probability of each. You win if you can correctly guess the color of the coin. While you are contemplating your choice, the game operator says, "Let me help you." He drops in two gold coins, shakes the pouch, then reaches in and draws out... two gold coins. What is the probability the remaining coin is gold?

**$\frac{3}{4}$** . *Label the three coins A, B, and C, with A being the coin that was originally in the bag. If A had been gold, the game operator could possibly have drawn AB, BC, or AC. If A had been silver, the operator could only have drawn BC. There are 4 equally likely possibilities, with 3 of them resulting in a gold coin remaining in the bag. **This is a variant of Lewis' Carroll's famous "Pillow Problem"... look it up!***

4. Fourteen people are in line at a cashier's office. Seven owe \$1.00, and the other seven are expecting a refund of \$1.00. Unfortunately, the cashier has an empty cash box, and cannot give refunds unless he receives money first. Given a random arrangement of people in line, what is the probability that the cashier can serve them all, in order, without ever dropping below \$0?

**$\frac{1}{8}$** ,  
or  
**12.5%**. *There are  $C(14, 7) = 3432$  ways for the 14 people to get in line. Let  $C_n$  denote the number of ways for  $n$  people who owe \$1 and  $n$  people who are owed \$1 to get in line without putting the cashier in debt. (Note that  $C_1 = 1$ , since if there is only one person of each type, the person who owes \$1 must be first in line.) Now recursively define  $C_n$  based on whether or not the cashier hits \$0 mid-line:  
 $C_2 = C_1^2 + C_1 = 2$ ,  $C_3 = C_1C_2 + C_1^2 + C_2 = 5$ ,  $C_4 = \dots 14$ ,  $C_5 = \dots 42$ ,  $C_6 = \dots 132$ ,  
 $C_7 = C_1C_6 + C_1C_5 + C_2C_4 + C_3^2 + C_4C_2 + C_5C_1 + C_6 = 429$ , so  $P = 429/3432 = 1/8$ .*



# Minnesota State High School Mathematics League Individual Event

## 2008-09 Event 5D SOLUTIONS

1. If  $\frac{3}{4}$  of 8 apples are worth exactly as much as  $\frac{7}{8}$  of 16 oranges, how many oranges can I get for the same amount of money that buys 75 apples?

**175**. *As silly as it may seem to compare apples & oranges, note that 6 apples are worth 14 oranges. Set up the proportion:  $\frac{6}{14} = \frac{75}{g} \Rightarrow \frac{3}{7} = \frac{75}{g}$ . Since  $3 \cdot 25 = 75$ ,  $g = 7 \cdot 25 = 175$ .*

2. The six faces of a cube are labeled with the six numbers 1, 2, 4, 8, 16, and 32, in some order. Three identical such cubes are placed on a table touching each other, as shown in Figure 2. Find the largest possible sum of the 11 visible faces.

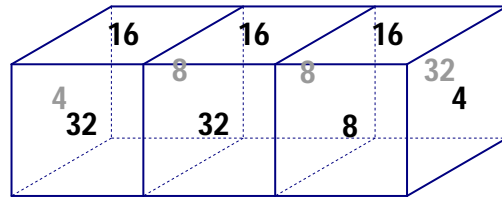
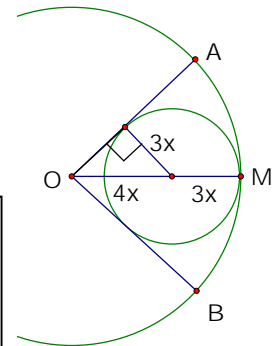


Figure 2

**176**. *As shown in the figure: the cubes can be placed so that the "1" face of each is against the table. The right-most cube is rotated 180° so that its "4" face (rather than "2") can face outwards. The sum is  $32(3) + 16(3) + 8(3) + 4(2) = 176$ .*

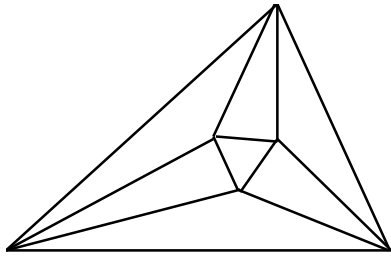
3. Points  $A$  and  $B$  are on a circle centered at  $O$ . A second circle is internally tangent to the first, and tangent to both  $\overline{OA}$  and  $\overline{OB}$  (Figure 3). If the ratio between the radii of the two circles is  $\frac{3}{7}$ , compute  $\cos \angle AOB$ .



$\cos \angle AOB = \frac{1}{8}$ .  $\sin \angle AOM = \frac{3x}{4x} = \frac{3}{4}$ , and by identity,  
 $\cos \angle AOB = \cos(2 \cdot \angle AOM) = 1 - 2 \sin^2 \angle AOM$   
 $= 1 - 2 \left(\frac{3}{4}\right)^2 = 1 - 2 \left(\frac{9}{16}\right) = 1 - \frac{18}{16} = -\frac{1}{8}$ .

4. Let  $k = 2^{2009} + 2009^2$ . Compute the units digit (ones' place) of  $k^{2009} + 2009^k$ .

**2**. *The units digit of the powers of 2 repeat in the cycle 2, 4, 8, 6, ... Since  $2009 \equiv 1 \pmod{4}$ , and  $2009^2$  ends in a 1,  $k$ 's units digit is 3. Now consider  $(\dots 3)^{2009} + 2009^{(\dots 3)}$ . The units digit of the powers of 3 repeat in the cycle 3, 9, 7, 1, ... Since  $2009 \equiv 1 \pmod{4}$ ,  $(\dots 3)^{2009}$  ends in a 3. The units digit of  $2009^k$  is 9, since  $k$  is odd.  $3 + 9$  ends in a 2.*



# Minnesota State High School Mathematics League

Team Event

## 2008-09 Meet 5 SOLUTIONS

1.  $n$  is a five-digit integer in which the first and last digits are unequal.  $m$  is the five-digit integer obtained by writing the digits of  $n$  in reverse order. (Neither integer begins with a zero.) If  $D = m - n$ , and the last three digits of  $D$  are  $\underline{029}$ , find  $D$ .

$D = \boxed{7029}$ .

2. In  $\triangle ABC$  (Figure 2),  $\overline{AB}$  and the perpendicular bisector of  $\overline{BC}$  meet at  $D$ , making  $AD = 9$  and  $DB = 7$ . It also happens that  $\overline{CD}$  bisects  $\angle ACB$ . Find the area of  $\triangle ACD$ .

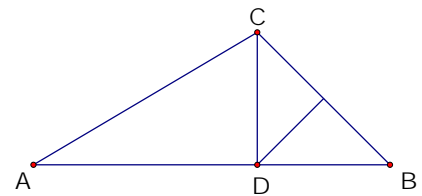


Figure 2

$Area(\triangle ACD) = \boxed{14\sqrt{5}}$ , or  $\boxed{31.305}$ .

3. How many distinct quadrilaterals exist within the MN State High School Math League logo at the top of this page?

$\boxed{15}$ .

4. Find the surface area of a cube whose volume is twice that of a cube with surface area 1. Express your answer in simplest radical form.

$\boxed{\sqrt[3]{4}}$ .

**Graders: note that simplest radical form is required!**

5. A rearrangement of the ten numbers 1, 1, 3, 3, 5, 5, 7, 7, 9, 9 is called "nerdy" if the four-digit sequence 1, 3, 3, 7 occurs left-to-right *somewhere* in the sequence (but not necessarily consecutively). Compute the number of nerdy rearrangements.

$\boxed{23940}$ .

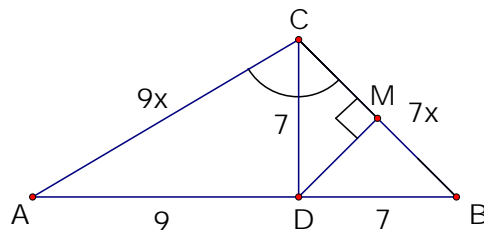
6. Right triangle  $PQR$  has perimeter 60. The altitude dropped from  $P$  to the hypotenuse has length 12. Find the area of  $\triangle PQR$ .

$Area(\triangle PQR) = \boxed{150}$ .

1. Compare this problem to #4 from Event A. Here, in the middle column,  $R - R = 0$  suggests that  $R$  is not involved in either lending or borrowing.  $T$  is still less than  $P$ , however, so  $T$  must borrow from  $S$ . Now  $(S - 1) - Q = 2 \Rightarrow S = Q + 3$ , and since  $Q$  is smaller than  $S$ ,  $Q$  must borrow from  $P$ .

$$\begin{array}{r} PQRST \\ -TSRQP \\ \hline xy029 \end{array} \Rightarrow \begin{array}{r} \overset{P-1}{\cancel{P}} \overset{10+Q}{\cancel{Q}} R \overset{S-1}{\cancel{S}} \overset{10+T}{\cancel{T}} \\ -T \quad S \quad RQ \quad P \\ \hline x \quad y \quad 0 \quad 2 \quad 9 \end{array} \Rightarrow \begin{array}{l} (10+Q) - S = (10+Q) - (Q+3) = 7 = y, \\ (10+T) - P = 9 \Rightarrow P - T = 1, \\ \text{so } (P-1) - T = x = 0. \end{array} \quad \mathbf{D = 7029}$$

2. By the Angle Bisector Theorem,  $\frac{AC}{BC} = \frac{9}{7}$ , so let  $AC = 9x$ ,  $BC = 7x$ . Since  $D$  lies on the perpendicular bisector of  $\overline{BC}$ ,  $DC = BD = 7$ . Let  $M$  be the midpoint of  $\overline{BC}$ . Then in  $\triangle CDM$ ,  $\cos \angle DCM = \frac{CM}{CD} = \frac{7x/2}{7} = \frac{x}{2}$ , and applying



the Law of Cosines to  $\triangle ACD$ ,  $9^2 = (9x)^2 + 49 - 2(9x)(7) \cos \angle ACD$

$$\Rightarrow 81 = 81x^2 + 49 - 126x \left( \frac{x}{2} \right) \Rightarrow 32 = 18x^2 \Rightarrow x = \frac{4}{3} \Rightarrow AC = 12.$$

Finally, by Heron's Formula,  $s = \frac{12+7+9}{2} = 14 \Rightarrow \text{Area}(\triangle ACD) = \sqrt{(14)(2)(7)(5)} = \mathbf{14\sqrt{5}}$

**Credit: 2002 AMC-12, problem #23**

3. Number the individual (small) triangles as shown in the diagram. For starters, each of triangles #2, #4, and #6 are adjacent to three other triangles, making  $3 \times 3 = 9$  quadrilaterals formed by pairs of adjacent triangles. (Yes, some of these are concave quadrilaterals, and they are valid!)

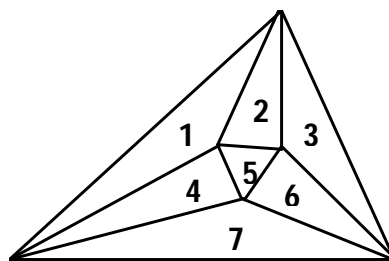
Then, count the following additional quadrilaterals:

$$(\#1 + \#2 + \#4 + \#5) \quad (\#2 + \#3 + \#5 + \#6)$$

$$(\#4 + \#5 + \#6 + \#7) \quad (\text{all but } \#1)$$

$$(\text{all but } \#3) \quad (\text{all but } \#7)$$

This makes a total of  $9 + 3 + 3 = \mathbf{15}$  quadrilaterals.



4. Since the ratio of volumes is the cube of the ratio of side lengths, if the volume is increasing by a factor of two, then the side lengths are increasing by a factor of  $\sqrt[3]{2}$ . Similarly (no pun intended), the ratio of surface areas is the square of the ratio of side lengths, which is  $(\sqrt[3]{2})^2 = \sqrt[3]{2^2} = \mathbf{\sqrt[3]{4}}$ .

5. There are  $\binom{10}{2}\binom{8}{2} = 1260$  ways to choose where the 5's and 9's will go, and these choices will have no effect on the "nerdiness" of the arrangement. Of the remaining 6 open spots (name the spots using the letters A through F), we may not place a 3 in spots A or F. This restricts the 3's to spots B, C, D, and E, leaving us  $\binom{4}{2} = 6$  possible cases for the placement of 3's:

**CD.** Then of the  $\binom{4}{2} = 6$  ways to place the 1's and 7's in spots A, B, E, F, only the arrangement 7, 7, 3, 3, 1, 1 fails to be nerdy. **5 nerdy arrangements here.**

**BD.** This forces us to place a 1 in spot A. Because at least one 7 will end up to the right of the second 3 in spot D, all **3 arrangements** of placing the remaining 1 and 7's **are nerdy.**

**CE.** This case is symmetric to **BD**, resulting in another **3 nerdy arrangements.**

**BC.** Again, a 1 goes at A, and similar to **BD**, all **3 arrangements are nerdy.**

**DE.** Symmetric to **BC**. **3 nerdy arrangements.**

**BE.** Put a 1 in spot A, a 7 in spot F, leaving **2 arrangements** for the 1 and 7 in spots C & D.

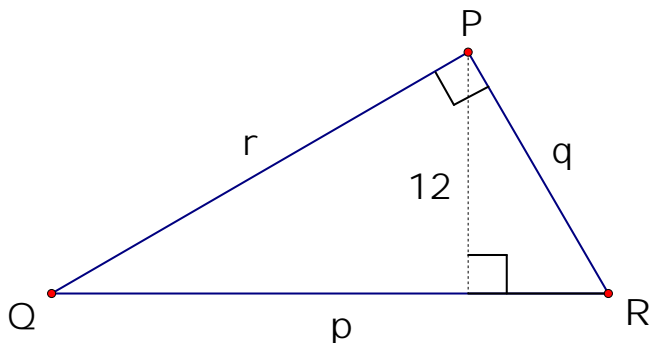
In all, we have  $1260(5 + 3 + 3 + 3 + 3 + 2) = \boxed{23940}$  nerdy rearrangements.

6. Refer to the diagram. Using  $\frac{1}{2}bh$ , we can

$$\text{express } \text{Area}(\triangle PQR) = \frac{1}{2}(12)(p) = 6p,$$

$$\text{or, using } \overline{PR} \text{ as the base, } \text{Area} = \frac{qr}{2}.$$

$$\text{So } 6p = \frac{qr}{2} \Rightarrow qr = 12p.$$



$$\text{Perimeter} = p + q + r = 60 \Rightarrow q + r = 60 - p. \text{ Squaring, } q^2 + 2qr + r^2 = 3600 - 120p + p^2.$$

$$\text{Substituting using } qr = 12p \text{ and } q^2 + r^2 = p^2: \quad p^2 + 24p = 3600 - 120p + p^2$$

$$\text{Solving for } p, 144p = 3600 \Rightarrow p = 25, \text{ and } \text{Area}(\triangle PQR) = 6p = \boxed{150}.$$

**Credit: Krantz, *Techniques of Problem Solving*, 1991, AMS, p. 122 #6**