Minnesota State High School Mathematics League

Individual Event

2008-09 Event 3A

SOLUTIONS

1. Solve for x : $\begin{cases} x+y=11 \\ 2x+y=50 \end{cases}$

Subtracting the first equation from the second yields x = 39 immediately.

***x* =** $\boxed{39}$.

2. *Definition: A lattice point is a point in the xy-plane with integer coordinates.*

Find the only lattice point in the 1st quadrant (and therefore, not lying on an axis) which is a solution of the system:

***(x, y)* =** $\boxed{(5,1)}$.

$$\begin{cases} y < \frac{1}{2}x - 1 \\ y < \frac{-2}{3}x + 5 \end{cases}$$

$$\Rightarrow \begin{cases} 1 < \frac{1}{2}x - 1 \\ 1 < \frac{-2}{3}x + 5 \end{cases} \Rightarrow \begin{cases} 2 < \frac{1}{2}x \\ -4 < \frac{-2}{3}x \end{cases} \Rightarrow \begin{cases} 4 < x \\ 6 > x \end{cases} \Rightarrow x = 5$$

Certainly this could be done with a careful graph, but it may be faster to realize that if there is only one solution point, and both inequalities are of the form y < , then the lattice point must be of the form (x, 1).

3. If $\begin{vmatrix} n & 2n & 3n \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 6$, find n.

Expanding the determinant along row 3, we have

$$(-1)^{3+2} \cdot 3 \cdot \begin{vmatrix} n & 3n \\ 1 & 2 \end{vmatrix} = 6 \Rightarrow -3 \cdot (2n - 3n) = 6 \Rightarrow 3n = 6$$

***n* =** $\boxed{2}$ **.**

4. ShaKiela and Wei-Chi share a birthday today. In three years, ShaKiela will be four times as old as Wei-Chi was when ShaKiela was two years older than Wei-Chi is today. If Wei-Chi is a teenager, find ShaKiela's age.

$\boxed{25}$ **.**

Suppose that "...when ShaKiela was two years older..." was x years ago. Then:

$$\begin{cases} 3\,\textit{years hence}: \; S+3 = 4(W-x) \\ x\;\textit{years ago}: \quad S - x = W + 2 \end{cases} \Rightarrow \begin{cases} \frac{S+3}{4} = W - x \\ S - 2 = W + x \end{cases} \Rightarrow \frac{5S}{4} - \frac{5}{4} = \frac{5}{4}(S-1) = 2W$$

Multiplying by $\frac{4}{5}$*, we see W is divisible by 5, and since Wei-Chi is a teenager, W = 15.* $\frac{5}{4}(S-1) = 2W \Rightarrow S - 1 = \frac{4}{5}(2 \cdot 15) \Rightarrow S = 25.$

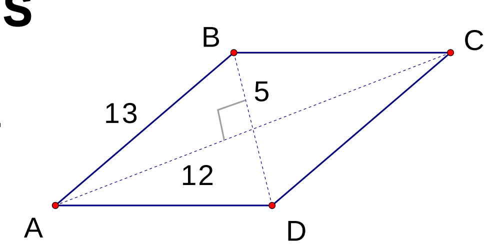
Minnesota State High School Mathematics League

Individual Event

2008-09 Event 3B

SOLUTIONS

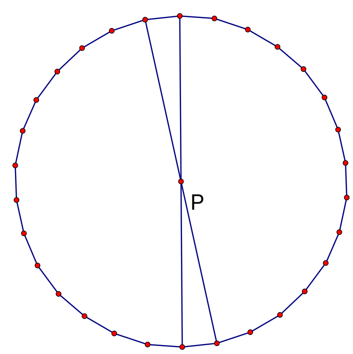
1. Rhombus *ABCD (Figure 1)* has sides of length 13. One of its diagonals has length 10. Find the area of *ABCD*.



$\boxed{120}$.

The diagonals of a rhombus bisect each other and meet at right angles. Thus, ABCD is made of four 5-12-13 triangles, each of area $\frac{1}{2}(5)(12) = 30$.

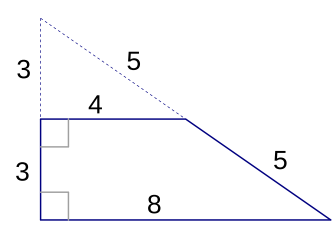
2. Two distinct diagonals are drawn inside a regular 30-gon. They intersect at the center of the polygon, *P*, and form both acute and obtuse angles at *P*. What is the largest possible degree measure of the obtuse angle?



$\boxed{168°}$.

The largest possible obtuse angle is created by two adjacent diagonals. These intersect at an acute angle of $360° \div 30 = 12°$. *Then* $180° - 12° = 168°$.

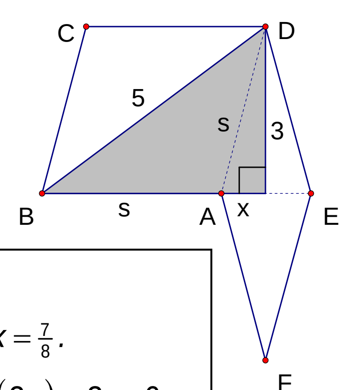
3. A trapezoid has bases of lengths 4 and 8, and sides of lengths 3 and 5. Find the area of the trapezoid.



$\boxed{18}$.

Extending the sides of the trapezoid, we find that the top base is the midline of a 6-8-10 (right) triangle.
Area(trapezoid) = $\frac{1}{2}(6)(8) - \frac{1}{2}(3)(4) = 18$.

4. Concave hexagon *ABCDEF* is formed by attaching rhombi *ABCD* and *ADEF* along edge $\overline{AD}$. Given that *A* lies on $\overline{BE}$, *BD* = 5, and *DF* = 6, find the area of *ABCDEF*.



$\boxed{\frac{117}{8}}$, *or* $\boxed{14.625}$.

The shaded triangle is 3-4-5, so $s + x = 4$.

$x^2 + 3^2 = s^2 \Rightarrow (s-4)^2 + 9 = s^2 \Rightarrow s = \frac{25}{8}$, $x = \frac{7}{8}$.

[CDEB] + [AEF] = $\frac{1}{2}(s + (s+2x))(3) + \frac{1}{2}(3)(2x) = 3s + 6x$.

Minnesota State High School Mathematics League

Individual Event

2008-09 Event 3C

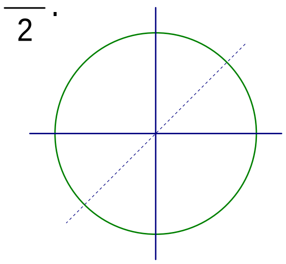
SOLUTIONS

1. Find the solution of the equation $\cos x - \sin x = 0$ where $\pi \le x < \frac{3\pi}{2}$.

$\boxed{\frac{5\pi}{4}}$, ***or*** $\boxed{3.927}$.

Graders: <u>Must</u> be in radians!

Adding sin *x* *to both sides yields* cos *x* = sin *x*. *This certainly occurs at* $x = \frac{\pi}{4}$, *but this is not in the desired domain. Reflect across the origin.*



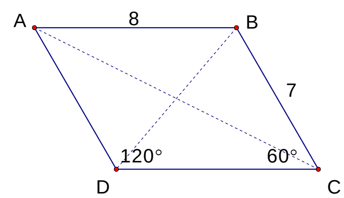
2. Find all solutions to the equation $\sec^2\theta - 3\sec\theta - 2 = 0$ on the interval $0 \le \theta < 2\pi$.

$\boxed{\theta \in \{1.286,\ 4.997\}}$

Graders: 1 pt. per correct <u>radian</u> value

Let $y = \sec\theta$. *Then* $y^2 - 3y - 2 = 0$. *By the Quadratic Formula,* $y \in \{-0.56155,\ 3.56155\}$. *Since* $\sec\theta$ *must be* ≤ -1 *or* ≥ 1, *we exclude* -0.56155. $\sec\theta \approx 3.56155 \Rightarrow \cos\theta \approx 0.28078$.

3. In parallelogram *ABCD*, $\overline{AC}$ and $\overline{BD}$ are diagonals. If *BC* = 7, *AB* = 8, and $m\angle C = 60°$, find the value of $AC^2 - BD^2$.



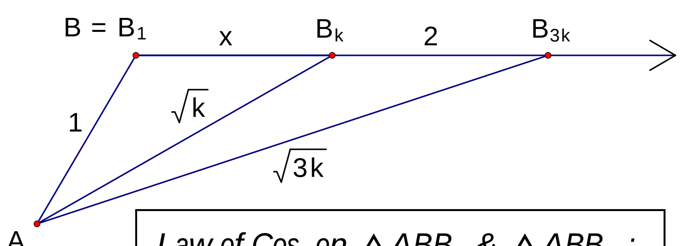
$\boxed{112}$.

Using the Law of Cosines on triangles ADC and CBD:

$AC^2 = 7^2 + 8^2 - 2(7)(8)\cos 120°$ $BD^2 = 7^2 + 8^2 - 2(7)(8)\cos 60°$

Subtracting, $AC^2 - BD^2 = -112(\cos 120° - \cos 60°)$.

4. In *Figure 4*, $m\angle ABC = 120°$ and *AB* = 1. Lying on side $\overline{BC}$ are the points B_1, B_2, B_3, ..., B_n such that the distance from *A* to B_k is $\sqrt{k}$. If the distance from B_k to B_{3k} is 2, find *k*.



$\boxed{7}$.

Law of Cos. on $\triangle ABB_k$ & $\triangle ABB_{3k}$:

$k = 1 + x^2 - 2x\cos 120° = 1 + x + x^2$

$3k = 1 + (x+2)^2 - 2(x+2)\cos 120°$

Equating at k, $x = 2 \Rightarrow k = 7$.

Questions 3 & 4 are courtesy of <u>The New York City Contest Problem Book, 1975-1984.</u>

Minnesota State High School Mathematics League

Individual Event

2008-09 Event 3D SOLUTIONS

1. Given that $2\log_2 x = 6$, find x.

***x* =** $\boxed{8}$.

$$\boxed{2\log_2 x = 6 \;\Rightarrow\; \log_2 x = 3 \;\Rightarrow\; x = 2^3 = 8.}$$

2. If $\sqrt{x \cdot \sqrt[5]{x}} = x^h$, find h.

h* =** $\boxed{\frac{3}{5}}$ ***or $\boxed{0.6}$.

$$\boxed{\sqrt{x \cdot \sqrt[5]{x}} = \left(x \cdot x^{1/5}\right)^{1/2} = \left(x^{6/5}\right)^{1/2} = x^{3/5} = x^h.}$$

3. Suppose $a = \log 5$, and $b = \log 9$, where the logarithms are base 10. Find $\log 12$ in terms of a and b.

log 12 = $\boxed{2 - 2a + \frac{b}{2}}$ ***or*** $\boxed{\frac{4 - 4a + b}{2}}$.

$$\log 10 = 1 = \log 5 + \log 2 \;\Rightarrow\; \log 2 = 1 - a$$

$$b = \log 9 = \log 3^2 = 2\log 3 \;\Rightarrow\; \log 3 = \frac{b}{2}$$

$$\log 12 = \log\left(2^2 \cdot 3\right) = 2\log 2 + \log 3 = 2(1 - a) + \frac{b}{2}$$

4. Find the sum of all positive integers N for which $\left[\sqrt{9 + \sqrt{N}} - \sqrt{9 - \sqrt{N}}\right]^2$ is an integer.

Applying the pattern $(x - y)^2 = x^2 - 2xy + y^2$ *to the term in brackets:*

$$\left[\sqrt{9 + \sqrt{N}} - \sqrt{9 - \sqrt{N}}\right]^2 = \left(9 + \sqrt{N}\right) - 2\sqrt{81 - N} + \left(9 - \sqrt{N}\right)$$

$$= 18 - 2\sqrt{81 - N}$$

We wish $81 - N$ *to be a perfect square. The values of N that do this are N = 80, 77, 72, 65, 56, 45, 32, 17. Their sum is 525.*

Minnesota State High School Mathematics League

Team Event

2008-09 Meet 3
SOLUTIONS

1. Convex pentagon *ABCDE* is inscribed in a circle of radius 1. *AB* = *BC*, *CD* = *DE* = *EA*, and *AC* = 2. Find *BD*.

BD* =** $\boxed{\dfrac{\sqrt{6}+\sqrt{2}}{2}}$ ***or $\boxed{1.932}$.

2. Given that $\log_{12} 3 = x$ and $\log_{12} 75 = y$, find $\log_{12} \dfrac{40}{9}$ in terms of *x* and *y*.

$\boxed{\dfrac{3+y-8x}{2}}$

3. Lines ℓ_1, ℓ_2, and ℓ_3 create $\triangle ABC$ *(Figure 3)*. The incircle has center *I*. A circle tangent to all three lines, but on the other side of ℓ_1 from *C*, has center *K* and radius 12. Given that *AK* = 13 and *BK* = 15, find *KI*.

KI* =** $\boxed{\dfrac{65}{4}}$ ***or $\boxed{16.25}$.

4. Four positive integers sum to 125. If you increase the first of these numbers by 4, decrease the second by 4, multiply the third by 4, and divide the fourth by 4, you produce four equal numbers. Find the four integers, listing them in the order presented in the problem.

$\boxed{\{16, 24, 5, 80\}}$

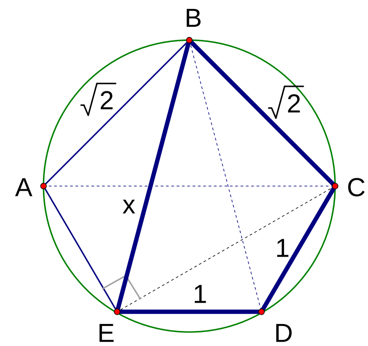
5. In parallelogram *ABCD (Figure 5)*, the bisector of $\angle ABC$ intersects $\overline{AD}$ at point *P*. If *PD* = 5, *BP* = 6, and *CP* = 6, find *AB*.

***AB* =** $\boxed{4}$.

6. If $\left[\log_2(4)-1\right]+\left[\log_2(6)-1\right]+\left[\log_2(8)-1\right]+\ldots+\left[\log_2(2008)-1\right]=\log_2(k!)$, find *k*.

***k* =** $\boxed{1004}$.

1. *Since the circle's radius is 1, AC = 2 requires that AC be a diameter. Using properties of 45-45-90 and equilateral triangles, we conclude that* $AB = BC = \sqrt{2}$ *and CD = DE = EA = 1. Also,* $CE = \sqrt{3}$, *since ACE is a right triangle. Let BD = BE = x. By Ptolemy's Theorem on quadrilateral BCDE,*



$$(1\cdot x)+\left(1\cdot\sqrt{2}\right)=\left(x\cdot\sqrt{3}\right) \Rightarrow \sqrt{2}=x\left(\sqrt{3}-1\right) \Rightarrow x=\boxed{\frac{\sqrt{6}+\sqrt{2}}{2}}$$

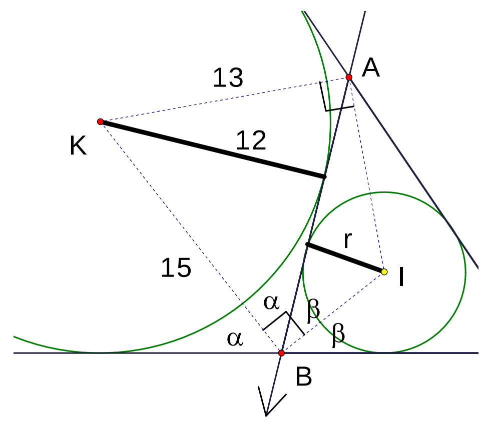
2. $\log_{12} 75 = \log_{12}\left(5^2\cdot 3\right) = 2\log_{12} 5 + \log_{12} 3$, *so* $\log_{12} 5 = \frac{y-x}{2}$.

$\log_{12} 4 = \log_{12}\frac{12}{3} = \log_{12} 12 - \log_{12} 3 = 1 - x$. $\quad\log_{12} 2 = \log_{12} 4^{1/2} = \frac{1}{2}\log_{12} 4 = \frac{1-x}{2}$.

$$\log_{12}\frac{40}{9} = \log_{12} 8 + \log_{12} 5 - \log_{12} 9 = 3\log_{12} 2 + \log_{12} 5 - 2\log_{12} 3 = 3\left(\frac{1-x}{2}\right) + \frac{y-x}{2} - 2x$$

$$= \boxed{\frac{3+y-8x}{2}}$$

3. *At point B,* $2\alpha + 2\beta = 180°$, *so* $\angle KBI$ *is a right angle. Similarly,* $\angle KAI$ *is also right. Dissecting* $\triangle BKA$ *using the radius of the large circle reveals both 5-12-13 and 9-12-15 right triangles. Dissection of* $\triangle BIA$ *(using the inradius) reveals similar triangles. AB = 5 + 9 = 14, but using proportionality, it also equals* $\frac{12}{5}r + \frac{12}{9}r = \frac{56}{15}r$,



thus $r = \frac{15}{4}$. $AI = \frac{13}{5}r = \frac{13}{5}\left(\frac{15}{4}\right) = \frac{39}{4}$, *and by the Pythagorean Theorem, KI =* $\boxed{\frac{65}{4}}$.

4. *Let A, B, C, and D be the original numbers.*

Then: $\begin{cases} A+B+C+D=125 \\ A+4=B-4=4C=\frac{D}{4} \end{cases}$

Using the equalities in the bottom line, $B = A+8$, $C = \frac{A}{4}+1$, $D = 4A+16$.

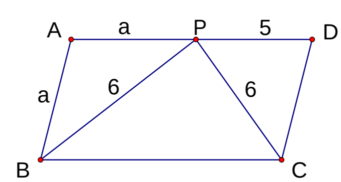
Substituting into the top line, $A+(A+8)+\left(\frac{A}{4}+1\right)+(4A+16) = \frac{25A}{4}+25 = 125 \Rightarrow A = 16$.

By back substitution, $B = A+8 = 24$, $C = \frac{A}{4}+1 = 5$, $D = 4A+16 = 80$. $\boxed{(16,24,5,80)}$.

5. *Observe that* $\triangle APB$ *and* $\triangle BPC$ *are similar isosceles triangles.*

So $\frac{AP}{PB} = \frac{PC}{BC} \Rightarrow \frac{a}{6} = \frac{6}{a+5} \Rightarrow a^2+5a=36 \Rightarrow a^2+5a-36=0$.

Factoring, $(a+9)(a-4)=0$, *and* $a = \boxed{4}$.



6. *Replacing all 1's with* $\log_2 2$, *the left side of the original equation becomes*

$\log_2 2 + \log_2 3 + \log_2 4 + \ldots + \log_2 1004 = \log_2(2\cdot 3\cdot 4\cdot\ldots\cdot 1004) = \log_2(1004!)$, *so k =* $\boxed{1004}$.