Minnesota State High School Mathematics League

Individual Event

2007-08 Event 2A

SOLUTIONS

1. $\boxed{\frac{54}{11}}$ $27 ‡ 6 = \frac{27\cdot 6}{27+6} = \frac{27\cdot \cancel{6}^{2}}{\cancel{33}_{11}} = \frac{54}{11}$

2. $\boxed{T = 0.15\cdot I - 374}$

Income(I)	Tax(T)
24,000	3226
24,100	3241
24,200	3256
24,300	3271

rate of change = $\frac{3241-3226}{24{,}100-24{,}000} = \frac{15}{100} = 0.15$

Using the point-slope method,

$(T-3226) = 0.15(I-24000)$

$T-3226 = 0.15I-3600 \quad \Leftrightarrow \quad T = 0.15\cdot I - 374$

3. $\boxed{N = \frac{8}{11}R + \frac{36}{11}}$ Apply the operations to N, one at a time:

$$N \to (N-5) \to 3(N-5) \to 3(N-5)+6 \to 4[3(N-5)+6]$$
$$\to 4[3(N-5)+6]-N \to 0.125\left(4[3(N-5)+6]-N\right)$$

Now simplify the final expression:

$$0.125\left(4[3(N-5)+6]-N\right) = 0.125\left(4[3N-9]-N\right)$$
$$= 0.125(11N-36)$$
$$= \frac{11N-36}{8} = R$$

Solving for N, we obtain

$$N = \frac{8}{11}R + \frac{36}{11}$$

4. $\boxed{m=15,\ n=21}$ (m and n can, of course, be interchanged in the solution.)

If GCD = 1, then LCM = 103, which is prime and cannot be the product of 4 primes.

If GCD = 2, then LCM = 104 = (2)(2)(2)(13), but since (GCD)(LCM) = (m)(n), one of the two integers must have more than "exactly two" prime factors.

If GCD = 3, then LCM = 105 = (3)(5)(7), leading to the solution $\boxed{m=15,\ n=21}$.

How can we prove that this is the <u>only</u> answer? Watch the League Notes for a proof...

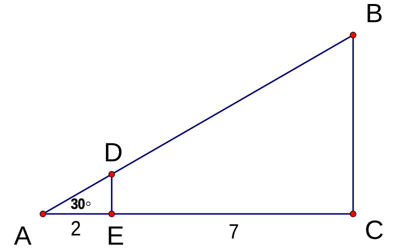
Minnesota State High School Mathematics League

Individual Event

2007-08 Event 2B

SOLUTIONS

1. $\boxed{3\sqrt{3}.}$ Since $AC = 9$, $BC = \frac{AC}{\sqrt{3}} = \frac{9}{\sqrt{3}} = 3\sqrt{3}$.



2. $\boxed{72.}$ Set each possible pair of sides equal, and solve for x:

$$3x+91=4x+19 \qquad 72=x$$

$$4x+19=5x+1 \qquad 18=x$$

$$3x+91=5x+1 \qquad x=45$$

Verify these values of x using the Triangle Inequality... then the largest x is 72.

3. $\boxed{7-2\sqrt{6}.}$ Apply the Pythagorean Theorem to right triangle *RNM:*

$$(4-x)^2+(3-x)^2=x^2$$

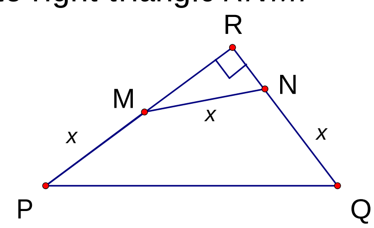
$$16-8x+x^2+9-6x+x^2=x^2$$

$$x^2-14x+25=0$$

$$x=\frac{14\pm\sqrt{14^2-4\cdot 25}}{2}$$

$$=7\pm\sqrt{7^2-5^2}$$

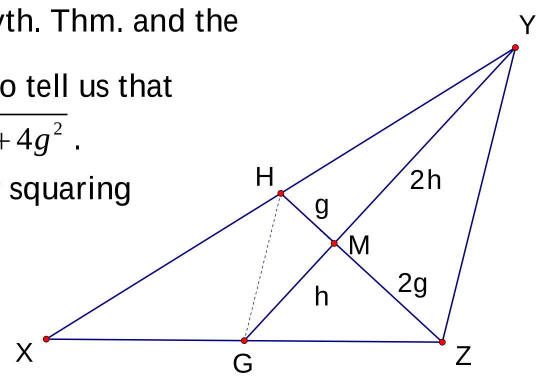
$$=7\pm 2\sqrt{6}$$



Since x is certainly less than 7, the only possibility is $7-2\sqrt{6}$.

4. $\boxed{\frac{\sqrt{5}}{4}}$ With the diagram labeled as shown, the Pyth. Thm. and the 2:1 ratio of intersecting medians combine to tell us that $YH = HX = \sqrt{g^2+4h^2}$, and $ZG = GX = \sqrt{h^2+4g^2}$.

The given ratio says that $4GX = 3HX$, which, after squaring both sides, simplifies to $g = \frac{2}{\sqrt{11}}h$.



$$\frac{GH}{HY} = \frac{\sqrt{g^2+h^2}}{\sqrt{g^2+4h^2}} = \frac{\sqrt{\frac{15}{11}h^2}}{\sqrt{\frac{48}{11}h^2}} = \frac{\sqrt{15}}{\sqrt{48}} = \frac{\sqrt{5}}{\sqrt{16}} = \frac{\sqrt{5}}{4}$$

Minnesota State High School Mathematics League

Individual Event

2007-08 Event 2C

SOLUTIONS

1. $\boxed{13^\circ.}$ Using the sine/cosine cofunction identity, $\sin(2x+8)=\cos(5x-9)$ is true when $(2x+8)=90-(5x-9)$. Solve for x to obtain 13°.

2. $\boxed{244^\circ.}$ Realizing that $-\sqrt{\dfrac{1+\sin 38^\circ}{2}}=-\sqrt{\dfrac{1+\cos 52^\circ}{2}}$, we recognize the half-angle formula for cosine. Since $+\sqrt{\dfrac{1+\cos 52^\circ}{2}}=\cos 26^\circ$, $-\sqrt{\dfrac{1+\cos 52^\circ}{2}}=-\cos 26^\circ$.
So $\sin x=-\cos 26^\circ=\cos 154^\circ=\sin(90-154)^\circ=\sin(-64)^\circ=\sin 296^\circ=\sin 244^\circ$, and the smallest positive x is 244°.

3. $\boxed{\left(\dfrac{90}{11}\right)^\circ}$ Using the given hint,

$$\sec 5x=\tan 3x+\tan 5x$$
$$\frac{1}{\cos 5x}=\frac{\sin 3x}{\cos 3x}+\frac{\sin 5x}{\cos 5x}$$

Multiplying both sides by $(\cos 3x)(\cos 5x)$ reveals the sum identity for sine:

$$\cos 3x=\sin 3x\cos 5x+\cos 3x\sin 5x=\sin 8x$$

So $3x+8x=90^\circ$, and $x=\left(\dfrac{90}{11}\right)^\circ$.

4. $\boxed{\left(\dfrac{180}{17}\right)^\circ}$ Multiply both sides by 16 sin x:

$$16(\cos 8x)(\cos 4x)(\cos 2x)(\cos x)(\sin x)=\sin x$$

Now, repeatedly, use factors of two along with the double angle formula for sine:

$$8(\cos 8x)(\cos 4x)\underline{(\cos 2x)(\sin 2x)}=4(\cos 8x)\underline{(\cos 4x)(\sin 4x)}=2(\cos 8x)(\sin 8x)$$

So $\sin 16x=\sin x \;\Rightarrow\; 16x+x=180^\circ$, and $x=\left(\dfrac{180}{17}\right)^\circ$.

Minnesota State High School Mathematics League

Individual Event

2007-08 Event 2D

SOLUTIONS

1. $\boxed{-6.}$ $2y+x+3=0 \Leftrightarrow y=\frac{-1}{2}x+\frac{3}{2}$, which is a line with slope $\frac{-1}{2}$.

$3y+ax+2=0 \Leftrightarrow y=\frac{-a}{3}x-\frac{2}{3}$, which is a line with slope $\frac{-a}{3}$.

The lines are perpendicular, so $\left(\frac{-1}{2}\right)\left(\frac{-a}{3}\right)=-1$, and $a = -6$.

2. $\boxed{\left(\frac{r}{r^2+1},\frac{r}{r^2+1}\right)}$

First find the equation of the line through the two intercepts $(r,0)$ and $\left(0,\frac{1}{r}\right)$:

slope = $\frac{1/r}{-r}=\frac{-1}{r^2}$, so using slope-intercept form, $y=\frac{-1}{r^2}x+\frac{1}{r}$.

Now let $y=x$: $x=\frac{-1}{r^2}x+\frac{1}{r} \Leftrightarrow \left(1+\frac{1}{r^2}\right)x=\frac{1}{r} \Leftrightarrow \left(\frac{r^2+1}{r^2}\right)x=\frac{1}{r} \Leftrightarrow x=\frac{r}{r^2+1}$,

and the desired point of intersection is $(x,y)=\left(\frac{r}{r^2+1},\frac{r}{r^2+1}\right)$.

3. $\boxed{4015.}$ Three noncollinear points determine a circle, so we know that $(-2,-3)$, $(2,5)$, and $(2007,y)$ must be collinear. The midpoint of $(-2,-3)$ and $(2,5)$ is $(0,1)$, quickly defining the line $y=2x+1$. $\therefore$ the desired y equals $2(2007)+1=4015$.

4. $\boxed{5}$ and $\boxed{845.}$ Noting that the point of tangency must be $(h,0)$, forcing $r = k$, we write:

> *Graders: <u>No</u> partial credit; both answers must be present.*

$$\begin{cases}(1-h)^2+(9-r)^2=r^2\\(8-h)^2+(8-r)^2=r^2\end{cases} \Rightarrow \begin{cases}1-2h+h^2+81-18r=0\\64-16h+h^2+64-16r=0\end{cases}$$

Subtracting, $-46+14h-2r=0$, and $h=\frac{r+23}{7}$.

$$(1-h)^2+(9-r)^2=r^2 \Rightarrow \left[\frac{-1}{7}(r+16)\right]^2+81-18r=0 \Rightarrow r^2-850r+4225=0$$

Applying the quadratic formula, we find $r\in\{5,845\}$.

Minnesota State High School Mathematics League

Team Event

2007-08 Meet 2
SOLUTIONS

1. $\boxed{y=\frac{3}{4}x+1}$ or $\boxed{y=\frac{3}{4}x+11}$. The normal form of the given line is $\frac{3x-4y}{5}=-\frac{24}{5}$.

The desired line(s) are 4 units away along the normal: $\frac{3x-4y}{5}=-\frac{24}{5}\pm 4$

Therefore, there are two possible parallel lines: $y=\frac{3}{4}x+1$ or $y=\frac{3}{4}x+11$.

2. $\boxed{110.}$ Given that we desire a solution in base *a*, it would be convenient to express the Pythagorean triples in terms of *a*:

a	3	5	7	9
b	4	12	24	40
c	5	13	25	41
b+*c*	9	25	49	81

Noticing that $b+c=a^2$, we can soon see that $b=\frac{a^2-1}{2}$ and $c=\frac{a^2+1}{2}$.

The perimeter is $a+b+c=a+\frac{a^2-1}{2}+\frac{a^2+1}{2}=a^2+a$, which in base *a* always equals 110 .

3. $\boxed{17^\circ}$ and $\boxed{101^\circ}$. Using unit circle relationships, the system can be rewritten as follows:

Graders: No partial credit; both answers must be present.

$$\begin{cases} x\cdot\cos 11^\circ - y\cdot\sin 149^\circ = \sin\alpha \\ x\cdot\cos 101^\circ + y\cdot\cos 31^\circ = \cos\alpha \end{cases} \Rightarrow \begin{cases} x\cdot\cos 11^\circ - y\cdot\sin 31^\circ = \sin\alpha \\ x\cdot -\sin 11^\circ + y\cdot\cos 31^\circ = \cos\alpha \end{cases}$$

Using Cramer's Rule,

$$x=\frac{D_x}{D}=\frac{\begin{vmatrix}\sin\alpha & -\sin 31^\circ \\ \cos\alpha & \cos 31^\circ\end{vmatrix}}{\begin{vmatrix}\cos 11^\circ & -\sin 31^\circ \\ -\sin 11^\circ & \cos 31^\circ\end{vmatrix}}=\frac{\sin\alpha\cos 31^\circ+\cos\alpha\sin 31^\circ}{\cos 11^\circ\cos 31^\circ-\sin 11^\circ\sin 31^\circ}=\frac{\sin(31+\alpha)^\circ}{\cos(31+11)^\circ}$$

Since we wish *x* to equal 1, we need $\sin(31+\alpha)^\circ=\cos 42^\circ=\sin 48^\circ$ *or* $\sin 132^\circ$, which occurs when $\boxed{\alpha\in\{17^\circ,101^\circ\}}$.

4. $\boxed{(0,2)}$ $\boxed{\left(-\frac{3}{4},\frac{5}{4}\right)}$ $\boxed{(1,1)}$ $\boxed{\left(\frac{13}{16},\frac{25}{16}\right)}$

Graders: *Give 1 point for each correct ordered pair.*

Each graph is comprised of the union of points on two intersecting lines.

$(x-y+2)(3x+y-4)=0$ involves the lines $y=x+2$ and $y=-3x+4$, while

$(x+y-2)(x-5y+7)=0$ involves the lines $y=-x+2$ and $y=\frac{1}{5}x+\frac{7}{5}$.

There will be four points of intersection, created by pairing one line from each graph:

$$y=x+2 \text{ and } y=-x+2 \Rightarrow x+2=-x+2 \Rightarrow (x,y)=(0,2)$$
$$y=x+2 \text{ and } y=\tfrac{1}{5}x+\tfrac{7}{5} \Rightarrow x+2=\tfrac{1}{5}x+\tfrac{7}{5} \Rightarrow (x,y)=\left(-\tfrac{3}{4},\tfrac{5}{4}\right)$$
$$y=-3x+4 \text{ and } y=-x+2 \Rightarrow -3x+4=-x+2 \Rightarrow (x,y)=(1,1)$$
$$y=-3x+4 \text{ and } y=\tfrac{1}{5}x+\tfrac{7}{5} \Rightarrow -3x+4=\tfrac{1}{5}x+\tfrac{7}{5} \Rightarrow (x,y)=\left(\tfrac{13}{16},\tfrac{25}{16}\right)$$

5. $\boxed{x=(a+b)-\sqrt{2ab}.}$

A similar solution method to that used in problem 3 of Event B, beginning with $\triangle RNX$:

$$(a-x)^2+(b-x)^2=x^2$$
$$a^2-2ax+x^2+b^2-2bx+x^2=x^2$$
$$x^2-2(a+b)x+\left(a^2+b^2\right)=0$$

$$x=\frac{2(a+b)\pm\sqrt{4(a+b)^2-4c^2}}{2}$$
$$=(a+b)\pm\sqrt{a^2+2ab+b^2-c^2}$$
$$=(a+b)\pm\sqrt{2ab}$$

Since $x<a$, it cannot be greater than $(a+b)$. Therefore, $x=(a+b)-\sqrt{2ab}$.

(Note that even this value of x is too big when beginning with a 5, 12, 13 right triangle. Can you find the class of right triangles for which x exists?)

6. $\boxed{45.}$ Using the formula *P* = *rt*, we generate the following table:

	P (papers)	*r (papers/min)*	*t (minutes)*
Sarah	60	*S*	90
Ryan	35	*R*	*X*
together	91	*S* + *R*	63

Since $60=S\cdot 90$, $S=\dfrac{2}{3}$. Then $91=(S+R)63 \Rightarrow S+R=\dfrac{13}{9}$, and $R=\dfrac{7}{9}$.

Finally, $35=RX=\dfrac{7}{9}X$, and $X=35\cdot\dfrac{9}{7}$, or 45 minutes.