# Minnesota State High School Mathematics League

## Individual Event

## 2007-08 Event 5A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have **20** minutes for this event.

9 1. In a weight training program, a woman begins with a weight of $w$, and then, increasing in equal increments, she works her way up to $x$, then $y$, then $z$, and finally to a top weight of $t$. If $w+x+y+z+t=45$, what is $y$?

Sid 2. Four men were being questioned by the police about a robbery.

"Jack did it," said Alan
"George did it,' said Jack.
"It wasn't me," said Sid.
"Jack is a liar if he said I did it," said George.

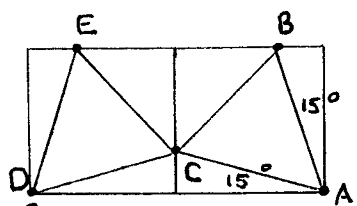
Only one had spoken the truth. Who committed the robbery?

7773331 3. The letters $A$, $B$, and $C$ represent distinct digits. $A$ is prime, $A-B=4$, and the seven digit number $AAABBBC$ is prime. What is that seven digit prime?

1.035 4. Rectangle $RSTU$ has sides of $RS=2$, $ST=1$. Five points placed interior to or on the boundary of this rectangle determine ten line segments. If the five points are placed so that the smallest of these ten segments is as large as possible, what will be the length of the shortest segment? Express your answer using decimal notation, rounded accurate to three places to the right of the decimal.

1. Let $c$ = the common increase. Then

$$\begin{aligned} w &= y-2c \\ x &= y-c \\ y &= y \\ z &= y+c \\ + \quad t &= y+2c \\ \hline 45 &= 5y;\ y=9 \end{aligned}$$




4. By the pigeon-hole principle, 3 of the points must be in one of the two 1×1 squares. In Team Event 4, #5, the solution for 3 points is as shown above for A, B, and C, where AB = 1.035. Locate D and E as shown.

2. If A did it, the truth was told by S, G. Similarly, $G \Rightarrow J, S$; $J \Rightarrow A, S, G$; $S \Rightarrow G$. Sid is, therefore, the guilty one.

3. Possibilities for $A-B = 5-1 = 4$, or $7-3$.

$C \notin \{0, 2, 4, 5, 6, 8\}$. Possibilities are, therefore, $A=5, B=1, C \in \{3, 7, 9\}$ or $A=7, B=3, C \in \{1, 9\}$

ARML Question, 1990

$5551113 \bmod 3 \equiv 0$ (ie 5551113 is divisible by 3)
$5551117 \bmod 3 \equiv 1$
$5551119 \bmod 3 \equiv 0$
$7773331 \bmod 3 \equiv 1$
$7773339 \bmod 3 \equiv 0$

The only candidates left

5551117 is divisible by 11.

$\therefore$ only 7773331 can be prime. It is!

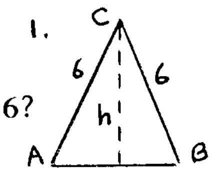
# Minnesota State High School Mathematics League

## Individual Event

## 2007-08 Event 5B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$8\sqrt{2}$ 1. What is the area of a $\Delta ABC$ that has sides of $AB = 4$, $BC = AC = 6$?

1. 


$h^2 + 2^2 = 6^2$

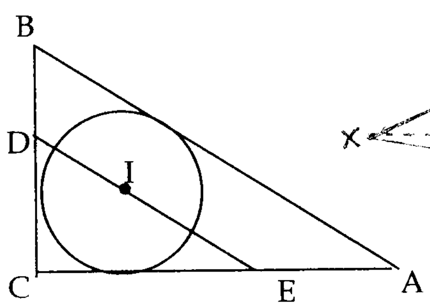
$h = 4\sqrt{2}$

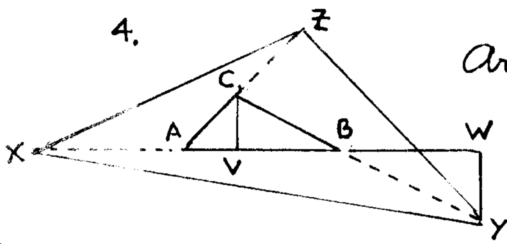
$\text{Area} = \frac{4}{2}(4\sqrt{2})$

$\pi - 2$ 2. The centers of a circle of radius 1 and an inscribed square coincide. What is the area of the region contained inside the circle, outside of the square?

$\frac{144\pi}{25}$ 3. In the right $\Delta ABC$ (Figure 3), $I$ is the center of the inscribed circle (the incenter). A line through $I$ parallel to the hypotenuse intersects $BC$ at $D$ and $AC$ at $E$, making $DI = 3$, $IE = 4$. Find the area of the circle.

$7a$ 4. Draw an arbitrary $\Delta ABC$. As in Figure 4, extend each of $AC$, $BA$, $CB$ to twice their length, forming $AZ = 2AC$, $BX = 2BA$, $CY = 2CB$. Express the area of $\Delta XYZ$ in terms of $a = Area(\Delta ABC)$. Hint: From C, drop altitude CV of $\Delta ABC$; then extend AB to W so that YW is an altitude of $\Delta XBY$




4. 


$\text{Area}(\Delta ABC) = \frac{1}{2}(AB)(CV)$

$= a$

$\Delta CVB \cong \Delta YWB$

(rt ∠s; vertical ∠s; CB = BY)

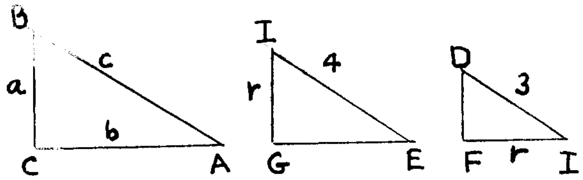
$\text{Area}(\Delta XBY) = \frac{1}{2}(XA + AB)(WY)$

Since $CV = WY$

$\text{Area}(\Delta XBY) = \frac{1}{2}(AB + AB)(CV) = 2a$

Similarly, $\text{Area}(\Delta YCZ) = \text{Area}(\Delta ZAX) = 2a$.

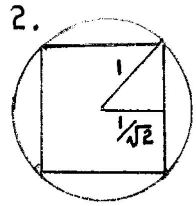
$\text{Area}(\Delta XYZ) = a + 2a + 2a + 2a = 7a$

3. 


$\frac{b}{c} = \frac{r}{3}$ and $\frac{a}{c} = \frac{r}{4}$

$c^2 = a^2 + b^2 = \left(\frac{rc}{4}\right)^2 + \left(\frac{rc}{3}\right)^2 = r^2c^2\,\frac{25}{144}$

$1 = \frac{25}{144}r^2$ ; $r = \frac{12}{5}$ ; $\text{Area} = \frac{144\pi}{25}$

2. 


$\pi(1)^2 - \left(\frac{2}{\sqrt{2}}\right)^2 = \pi - 2$

4. Possibilities after 2nd roll:

4,4 gives a Yahtzee, prob (Yahtzee) $= \frac{1}{36}$

prob (one 4) $= \frac{1}{6} \cdot \frac{5}{6} (2) = \frac{10}{36}$

prob (no 4's) $= \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36}$

Possibilities after 3rd roll

→ if starting with four 4's prob (Yahtzee) $= \frac{1}{6}$

→ if starting with three 4's the analysis is the same as that for the second roll.

So —

$$\text{prob (Yahtzee)} = \frac{1}{36} + \frac{10}{36} \cdot \frac{1}{6} + \frac{25}{36} \cdot \frac{1}{36} = \frac{36 + 60 + 25}{36^2} = \frac{121}{1296}$$

$$\text{prob (four 4's)} = \frac{10}{36} \cdot \frac{5}{6} + \frac{25}{36} \cdot \frac{10}{36} = \frac{300 + 250}{36^2} = \frac{550}{1296}$$

# 2007-08 Event 5C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

Though not necessary for answering the questions posed in #4, one gets a check on the work by noting that prob (three 4's) $= \frac{25}{36} \cdot \frac{25}{36}$ and $\frac{121 + 550 + 625}{1296} = 1$

4536 1. How many integers between 1000 and 9999 have distinct digits?

4.12 2. The value of \$500 ten years after having been invested at an annual interest rate of 4% is $\$500(1.04)^{10}$. If one only has at hand a cheap calculator that does not have a key for raising to powers, the value after ten years may be approximated by multiplying 500 by the first three terms of the expansion of $(1+.04)^{10}$. What error would be made (in dollars and cents) by using this estimate?

$\frac{3015}{4018}$ 3. Distinct integers $a$ and $b$ are chosen independently and at random from the set $\{0, 1, 2, \ldots, 2007, 2008\}$. What is the probability that $ab$ is even?

Graders: give 1 point for each correct answer

4. In the game of Yahtzee, five dice are thrown. On such a roll, Pat gets 4, 4, 4, 3, 5. The rules allow her two more chances to roll any subset of the five dice to try to achieve a goal. Hoping for four of a kind (four 4's in this case) or five of a kind (called a Yahtzee), she decides to save the first three 4's, and to roll the other two again. Her strategy is

- if she gets 4, 4, she will have a Yahtzee, and she will quit.
- If she gets just one 4, she will save it and roll the one other die for her third and final roll.
- if she gets no 4, she will roll the two non-4's for her third and final roll.

$\frac{550}{1296}$ (a) What is the probability that Pat will wind up with four of a kind?

$\frac{121}{1296}$ (b) What is the probability that Pat will wind up with a Yahtzee?

1. $\underline{9} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} = 4536$

2. $500(1+.04)^{10} \approx 500\left[1 + 10(.04) + \frac{10(9)}{2}(.04)^2\right] = 736.00$

$500(1.04)^{10} = 740.12$

Error $= \$4.12$

3. It is easier to find prob ($a \cdot b$ is odd) = prob (both $a$ and $b$ are odd).

prob ($a$ is odd) $= \frac{1004}{2009}$ ; prob ($b$ is odd) $= \frac{1003}{2008} = \frac{1003}{2(1004)}$

prob ($ab$ is even) $= 1 -$ prob ($ab$ is odd) $= 1 - \frac{1004}{2009} \cdot \frac{1003}{2(1004)} = \frac{3015}{4018}$

Solutions

4. Area$(\Delta ABC) = 70 = \frac{1}{2}(BC)(ht) = \frac{1}{2}\sqrt{4^2+2^2}(ht) = \sqrt{5}(ht)$, so $ht = \frac{70}{\sqrt{5}}$

The line through BC is, in normal form, $\frac{x+2y}{\sqrt{5}} = \frac{11}{\sqrt{5}}$ ← its distance from the origin. A must therefore lie on a line at a distance of $\frac{11}{\sqrt{5}} + \frac{70}{\sqrt{5}}$ from the origin, parallel to BC; i.e. on $\frac{x+2y}{\sqrt{5}} = \frac{81}{\sqrt{5}}$.

Choose A to be the point where this line intersects the $x$-axis: $A(81,0)$

# 2007-08 Event 5D

Questions in this event are written, with permission, as variations of problems from the 2006 AMC-12 Exam. Review of this exam is excellent preparation, not only for this event, but for the 2007 AMC-12, which we strongly encourage you to take.

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

\$ 52,500 ______ 1. John Q. Public inherited some money. He first gave a tithe (one tenth) of this money to his church. No tax is paid on such a donation. He then paid a 20% federal tax on what he had left, and finally, he paid the state tax of 10% of what was left after paying the federal tax. Taxes paid to the federal and state governments came to \$13,230 How much did he inherit?

Graders: Give 1 point for each correct answer

2. For each positive integer $n$, $S(n)$ is defined to be the sum of the digits of $n$. Find $n + S(n) + S(S(n))$ for

(a) 2010 ______ (a) $n = 1978$ (b) $n = 1980$

(b) 2007 ______

30 ______ 3. The set $\{4,7,12,17\}$ is augmented by a fifth element $n$, not equal to any of the other four. The median of the resulting set is equal to its mean. What is the sum of all possible values of $n$?

A(81, 0) ______ 4. $\Delta ABC$ has an area of 70. $B = (3,4)$ and $C = (7,2)$. Find the point $A(r,s)$ such that $r \geq 0$, $s \geq 0$, and $r + s$ is as large as possible.

1. A = amount inherited

$.9A$ = amount after tithe

$(.9A)(.2)$ = federal tax

$\{.9A - [(.9A)(.2)]\}(.1)$ = state tax

Total tax $= 13{,}500 = .18A + \{.09A - .018A\}$

Solving, $A = 52{,}500$

2. (a) $1978 + 25 + 7 = 2010$

(b) $1980 + 18 + 9 = 2007$

3. If $n < 7$, median $= 7$, and

$\frac{4+n+7+12+17}{5} = \frac{n+40}{5} = 7$

has no solution.

If $7 < n < 12$, median $= n$, and

$\frac{n+40}{5} = n$ has solution $n = 10$

If $n > 12$, median $= 12$, and

$\frac{n+40}{5} = 12$ has solution $n = 20$.

Sum $= 10 + 20 = 30$

Answers

# Minnesota State High School Mathematics League

## Team Event

### 2007-08 Meet 5

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

480 1. How many positive integers are composed of four distinct digits such that one digit is the average of the other three?

$53.1^\circ$ 2. In $\Delta ABC$, $\angle B = 90^\circ$ and $\sin A + \sin C = \frac{7}{5}$. Find to the nearest tenth of a degree the largest acute angle in $\Delta ABC$.

$\frac{864}{25}$ ← or 34.56

3. In the right $\Delta ABC$ (Figure 3), $I$ is the center of the inscribed circle (the incenter). A line through $I$ parallel to the hypotenuse intersects $BC$ at $D$ and $AC$ at $E$, making $DI = 3, IE = 4$. Find the area of $\Delta ABC$.

$\frac{a}{3}$ 4. $\Delta ABC$ is an arbitrary scalene triangle having medians that intersect at $M$ (Figure 4). Find the area of quadrilateral $CEMD$ in terms of $a = Area(\Delta ABC)$.

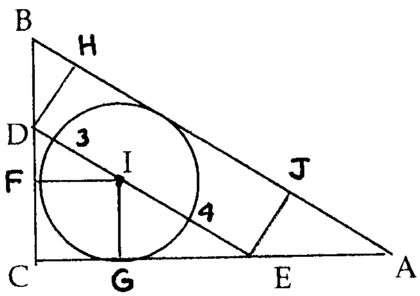



Figure 3

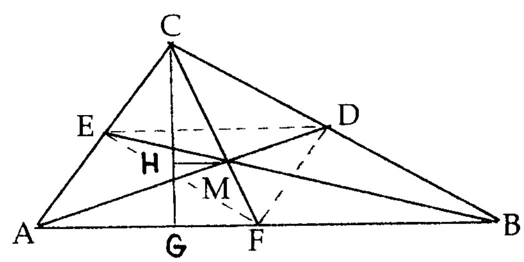



Figure 4

4. $\Delta DCE \cong \Delta EFD \cong \Delta FEA \cong \Delta BDF$ so $Area(\Delta DCE) = \frac{1}{4} Area(\Delta ABC) = \frac{a}{4}$

$\Delta CDE \sim \Delta ABC$ with common ratio $= \frac{1}{2}$; in particular $ED = \frac{1}{2} AB$

$\Delta DEM \sim \Delta\ BM$ and since $ED = \frac{1}{2} AB$, $ht(\Delta DEM) = \frac{1}{2} ht(\Delta ABM)$, so $Area(\Delta DEM) = \frac{1}{4} Area(\Delta ABM)$. $\frac{HG}{CG} = \frac{MF}{CF} = \frac{1}{3}$, so $HG = \frac{1}{3} CG$

$Area(\Delta ABM) = \frac{1}{2} HG \cdot AB = \frac{1}{3}\left(\frac{1}{2} CG \cdot AB\right) = \frac{1}{3} Area(\Delta ABC) = \frac{a}{3}$

$Area(DCEM) = a(\Delta DCE) + a(\Delta DEM) = \frac{a}{4} + \frac{1}{4}\left(\frac{a}{3}\right) = \frac{3+1}{12} = \frac{a}{3}$

Answers

# Minnesota State High School Mathematics League

## Team Event

## 2007-08 Meet 5

**CONTINUED**

5. On an $n \times n$ "checkerboard" of boxes, the integers from 1 through $n^2$ are entered in successive boxes, with the first $n$ integers placed consecutively in row one, the next $n$ integers placed in row two, etc. (Figure 5 shows the checkerboard for $n = 3$.) Suppose we choose $n$ of these integers in such a way that no two of them are in the same row or the same column. Independent of how this choice is made, it turns out that the sum is dependent only on the value of $n$. In fact, this sum is expressible as a polynomial in $n$. What is this polynomial?

$p(n) = \frac{n}{2}(n^2+1)$ or $\frac{1}{2}n^3 + \frac{1}{2}n$

6. In the game of Yahtzee, five dice are thrown. On such a roll, Mary gets 2, 3, 4, 3, 6. The rules allow her two more chances to roll any subset of the five dice to try to achieve a goal. Hoping for a large straight (a sequence of five consecutive integers) or at least a small straight (a sequence of four consecutive integers), she decides to save the first three dice, and to roll the other two again. Her strategy is
   - if she gets 1, 5 or 5, 6, she will have a large straight, and she will quit.
   - If she gets just one die that will extend her sequence of consecutive integers (that is a 1 or a 5), she will save it and roll the one other die for her third and final roll.
   - if she gets neither a 1 nor a 5, she will roll both dice again for her third and final roll.

$\frac{173}{648}$ (a) What is the probability that Mary will wind up with a large straight?

$\frac{347}{648}$ (b) What is the probability that Mary will wind up with a small straight?

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

Figure 5

Team________________________________

# Team Event 5 Solutions

1. The smallest possible average is $\frac{1+2+3}{3}=2$, and the largest is $\frac{7+8+9}{3}=8$
List all ways to get each possible average. Be systematic to get them all.

| Average | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| Possible Sums | $\boxed{015}$ | $\boxed{018}$ | $\boxed{039}$ | $\boxed{069}$ | $\boxed{189}$ | $\boxed{489}$ | ~~789~~ |
| | ~~024~~ | $\boxed{027}$ | ~~048~~ | $\boxed{078}$ | $\boxed{279}$ | ~~579~~ | |
| | ~~123~~ | ~~036~~ | $\boxed{057}$ | ~~159~~ | ~~369~~ | ~~678~~ | |
| | | $\boxed{045}$ | $\boxed{129}$ | $\boxed{168}$ | $\boxed{378}$ | | |
| | | $\boxed{126}$ | $\boxed{138}$ | $\boxed{249}$ | $\boxed{459}$ | | |
| | | ~~135~~ | ~~147~~ | ~~258~~ | ~~468~~ | | |
| | | ~~234~~ | $\boxed{156}$ | $\boxed{267}$ | ~~567~~ | | |
| | | | $\boxed{237}$ | $\boxed{348}$ | | | |
| | | | ~~246~~ | ~~357~~ | | | |
| | | | ~~345~~ | 456 | | | |
| | (1)+(0) | (3)+(1) | (2)+(4) | (2)+(4) | (0)+(4) | (0)+(1) | (0)+(0) |

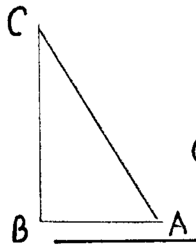
List systematically to get them all. Delete those using the digit at the top of the column.
Count in each column

(those involving zero) + (those not invol. zero)

Zero cannot be a lead digit. These can be used to form $3 \cdot 3 \cdot 2 \cdot 1 = 18$

These can be used to form $4 \cdot 3 \cdot 2 \cdot 1 = 24$

$$(8)\cdot 18 + (14)(24) = 480$$



2. Since $\sin A = \cos C$ and $\sin C = \cos A$,

$$\left.\begin{aligned}\sin A + \sin C &= \tfrac{7}{5}\\ \cos A + \cos C &= \tfrac{7}{5}\end{aligned}\right\}\ \text{Square both and add}$$

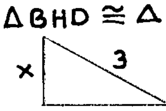
$$1 + 2\sin A \sin C + 2\cos A \cos C + 1 = 2\left(\tfrac{49}{25}\right)$$

$$\cos(A-C) = \tfrac{49}{25} - 1 = \tfrac{24}{25};\quad A-C = 16.26^\circ$$

Since $A+C=90^\circ$, $A = 53.13^\circ \approx 53.1^\circ$

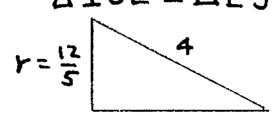
3. See Figure 3 on the Answer Sheet.
As in Event 5B, find $r=\frac{12}{5}$

$\Delta BHD \cong \Delta DFI$ $\qquad$ $\Delta IGE \cong \Delta EJA$



$r=\frac{12}{5}$

$x=\frac{9}{5}$



$y=\frac{16}{5}$

$$BC = BD + DF + FC = 3 + \tfrac{9}{5} + \tfrac{12}{5} = \tfrac{15+9+12}{5}$$

$$CA = CG + GE + EA = \tfrac{12}{5} + \tfrac{16}{5} + 4 = \tfrac{12+16+20}{5}$$

$$\text{Area}(\Delta ABC) = \tfrac{1}{2}\cdot\tfrac{36}{5}\cdot\tfrac{48}{5} = \tfrac{864}{25}$$

4. See solution on the Answer Sheet.

5. $\{1, 2, ---, n\}$ $\quad$ 1st row

$\{n+1, n+2, ---, n+n\}$ $\quad$ 2nd row

$\{2n+1, 2n+2, ---, 2n+n\}$ $\quad$ 3rd row

$\vdots$

$\{(n-1)n+1, (n-1)2+2, ---, (n-1)n+n\}$ $\quad$ nth row

Numbers chosen are $a_i n + b_j$ where
$a_i \in \{0, 1, --- n-1\} = \mathcal{R}$
$b_j \in \{1, 2, --- n\} = \mathcal{S}$

Since all members of $\mathcal{R}$ and all members of $\mathcal{S}$ got chosen, the sum, in some order, is

$$(0+1+---+n-1)n + (1+2+---+n) = p(n)$$

$$p(n) = \frac{(n-1)n}{2}n + \frac{n(n+1)}{2}$$

$$p(n) = \tfrac{1}{2}\left[n^3 - n^2 + n^2 + n\right] = \tfrac{n}{2}(n^2+1)$$

[NYSML 1989]

6. Possibilities after 2nd roll --- using a left and right die to depict all possibilities.

<u>Large Straight</u> $\boxed{1|5}$ $\boxed{5|1}$ $\boxed{5|6}$ $\boxed{6|5}$ prob(lg st) $=\frac{1}{9}$

<u>Adding one die to the sequence</u>

$\boxed{1|1}$ $\boxed{1|2}$ $\boxed{1|3}$ $\boxed{1|4}$ $\boxed{1|6}$ $\boxed{2|1}$ $\boxed{3|1}$ $\boxed{4|1}$ $\boxed{6|1}$

$\boxed{2|5}$ $\boxed{3|5}$ $\boxed{4|5}$ $\boxed{5|5}$ $\boxed{5|2}$ $\boxed{5|3}$ $\boxed{5|4}$

prob (1 2 3 4) $=\frac{1}{4}$ ; prob (2 3 4 5) $=\frac{7}{36}$

<u>Adding nothing to the sequence</u>

$\boxed{x|y}$ where $x \in \{2,3,4,6\}$, $y \in \{2,3,4,6\}$ prob $(3,4,5) = \frac{4}{9}$

Possibilities after 3rd roll:
- from 1 2 3 4 to 1 2 3 4 5 prob $=\frac{1}{6}$
- from 2 3 4 5 to $\left\{\begin{matrix}1\ 2\ 3\ 4\ 5\\ 2\ 3\ 4\ 5\ 6\end{matrix}\right\}$ prob $\frac{2}{6}$
- from 2 3 4, analysis is same as for the third role

$$\text{prob (lg st)} = \tfrac{1}{9} + \tfrac{1}{4}\cdot\tfrac{1}{6} + \tfrac{7}{36}\cdot\tfrac{2}{6} + \tfrac{4}{9}\cdot\tfrac{1}{9} = \tfrac{173}{648}$$

$$\text{prob (sm. st.)} = \tfrac{1}{4}\cdot\tfrac{5}{6} + \tfrac{7}{36}\cdot\tfrac{4}{6} + \tfrac{4}{9}\cdot\tfrac{4}{9} = \tfrac{347}{648}$$