Solutions

Minnesota State High School Mathematics League

Individual Event

2007-08 Event 1A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$m^2 n p$

1. If m, n, and p are distinct prime integers, what least common denominator should be used to add

$$\frac{1}{mn}+\frac{1}{m^2n}+\frac{1}{np}?$$

$\frac{17}{12}$

2. Express $.666....+\frac{1}{1.333....}$ as the quotient of two relatively prime integers.

2.7

3. My new pedometer, when strapped to my ankle, counts the number of steps I take, and then reports the miles I have walked by multiplying the number of steps by the length of a step – which I must enter. The length of step is to be entered as a decimal m.n where m is in feet, n in inches. I first entered 3.1 for a step of 37 inches, but that was evidently incorrect, because the pedometer recorded as 2.4 miles a distance known to be just 2 miles. What length of step m.n (remember that, for example, 2.9 means 2 feet, 9 inches) should I enter to get the best approximation to the correct distance of 2 miles? (There are 5280 feet to a mile.)

$x_1 = 2$

$x_2 = 3$

4. Find positive integers x_1 and x_2, $x_1 < x_2$ such that $\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_1 \cdot x_2}=1$

2. $\frac{2}{3}+\frac{1}{4/3}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}$

4 $x_2 + x_1 + 1 = x_1 x_2$

$x_2(x_1 - 1) = x_1 + 1$

$x_2 = \frac{x_1+1}{x_1-1} = 1 + \frac{2}{x_1-1}$

The only integer solution with $x_1 < x_2$ is $\begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases}$

(adapted from BMK)

3. Let m = number of steps I actually took.

Then in inches, $m(37) = 5280(2.4)(12)$

If s measures my actual step in inches,

$5280(2)(12) = sm = s\frac{(5280)(2.4)(12)}{37}$

$s = \frac{2(37)}{2.4} = 30.83.. \approx 31$ inches

s = 2 feet, 7 inches. Enter 2.7

OR more simply, play the percentages;

$2.4p = 2.0$ so $p = \frac{20}{24} = \frac{5}{6}$

$37(\frac{5}{6}) = 30.8$

Solutions

Minnesota State High School Mathematics League

Individual Event

2007-08 Event 1B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

75° 1. Figure 1 shows an equilateral ΔAEF inscribed in a square ABCD. Find the measure in degrees of $\angle AEB$.

72° 2. Figure 2 shows an equilateral ΔBGF inscribed in a regular pentagon ABCDE. Find the measure in degrees of $\angle DGF$.

20° 84° 3. In an isosceles ΔABC, $m(\angle B) = 7m(\angle A)$. Find two possible values for the measure of $\angle C$. (Give one point for each correct answer)

56° 4. In ΔABC (Figure 4), $BE = BF$, $CD = CF$, and $m(\angle A) = 68°$. Find $m(\angle EFD)$.

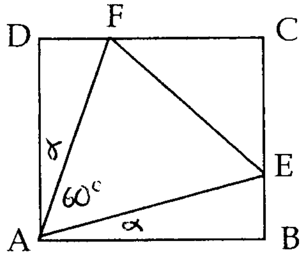


Figure 1

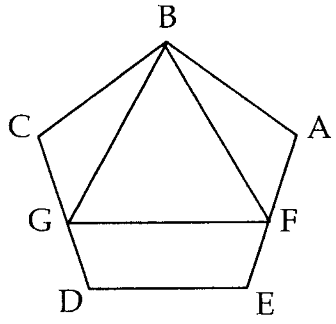


Figure 2

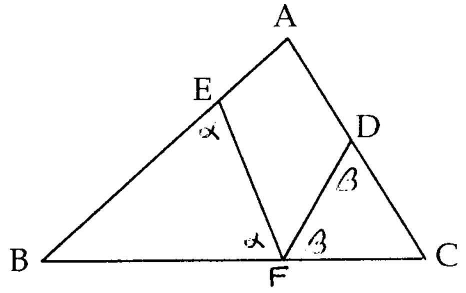


Figure 4

1. Let $\alpha = \angle BAE = \angle DAF$:

$2\alpha + 60 = 90$; $\alpha = 15$

$\angle AEB = 90 - 15 = 75°$

2. Vertex angles at D and E are $\frac{3(180)}{5} = 108$

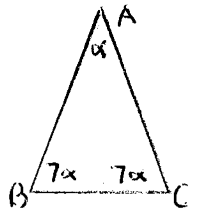
$2(\angle DGF) + 2(108) = 360$

$2(\angle DGF) = 144$

$\angle DGF = 72°$

[MML, Mar. 2006]

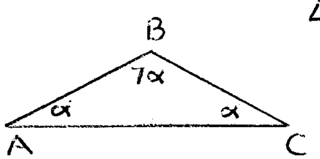
3.



Case I

$15\alpha = 180$

$\alpha = 12$; $\angle C = 84$



Case II

$9\alpha = 180$

$\alpha = 20$; $\angle C = 20°$

4. Let base angles of isosceles ΔBEF be α, those of ΔCDF be β.

$2\alpha + \angle B = 180$

$\underline{2\beta + \angle C = 180}$

$2(\alpha + \beta) + \angle B + \angle C = 360$

$\alpha + \beta = 180 - \frac{1}{2}(\angle B + \angle C)$

$\angle EFD = 180 - (\alpha + \beta) = \frac{1}{2}(\angle B + \angle C)$

But $\angle B + \angle C = 180 - \angle A = 180 - 68 = 112°$

$\angle EFD = \frac{1}{2}(112) = 56°$

Solutions

Minnesota State High School Mathematics League

Individual Event

2007-08 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{x}{5}$ 1. As point $P(x,y)$ moves through the second quadrant following the path of a circle of radius five, OP makes an angle of θ with the positive x-axis (Figure 1). Express $\cos\theta$ in terms of x.

$-\frac{\sqrt{25-y^2}}{5}$ 2. As point $P(x,y)$ moves through the second quadrant following the path of a circle of radius five, OP makes an angle of θ with the positive x-axis (Figure 1). Express $\cos\theta$ in terms of y.

120° 3. Suppose that in Figure 1, the line from $A(5,0)$ to P has length $5\sqrt{3}$. What is the measure of θ to the nearest degree?

188.447 4. Figure 4A comes from Peter Apianus, *Quadrans Astronomicus* (1532). It depicts two observers trying to determine the height of a tower. The information is shown more clearly in Figure 4B where $DB = 246$, $\angle ADC = 50°$, and $\angle DBC = 25°$. What is the height AC, correct to three places to the right of the decimal?

1. $\cos\theta = \frac{x}{5}$

2. $\cos\theta = -\frac{\sqrt{25-y^2}}{5}$

3. $(5-x)^2 + y^2 = 75$

$$25 - 10x + \underbrace{x^2 + y^2}_{25} = 75$$

$$-10x = 25$$

$$x = -\frac{25}{10} = -\frac{5}{2}$$

$$\cos\theta = \frac{-5/2}{5} = -\tfrac{1}{2}$$

$$\theta = 120°$$

4. Let $h = AC$.

$\tan 50° = \frac{h}{AD}$ so $AD = h\cot 50°$

$$\tan 25° = \frac{h}{AD + 246} = \frac{h}{h\cot 50° + 246}$$

$$h(\tan 25° \cot 50°) + 246\tan 25° = h$$

$$h = \frac{246\tan 25°}{1 - \tan 25° \cot 50°} = \frac{246}{\cot 25° - \cot 50°}$$

$$h = 188.447$$

Solutions

Minnesota State High School Mathematics League

Individual Event

2007-08 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$-1+2i$, $-1-2i$ 1. Find both roots of $(x+1)^2=-4$. Certainly accept $-1 \pm 2i$

-20 2. The polynomial function $f(x)$ has exactly three roots at $x=1$, $x=-\frac{4}{3}$, and $x=\frac{3}{2}$. $f(0)=-24$. Find $f(-1)$.

-4 3. A second degree polynomial function g(x) passes through the points $\left(\frac{3}{2},2\right)$, $(2,3)$, and $\left(\frac{5}{2},1\right)$. Find $g(3)$.

$-4x+5$ 4. The polynomial $P(x)$ has integer coefficients, and leaves a remainder of -3 when divided by $(x-2)$. The remainder is 17 when P(x) is divided by $(x+3)$. What is the remainder when P(x) is divided by $(x-2)(x+3)$?

1. $x+1 = \pm 2i$

$x = -1 \pm 2i$

2. [MML March 2006]

$f(x) = k(x-1)(3x+4)(2x-3)$

$f(0) = k(-1)(4)(-3) = -24$

$\therefore\ k = -2$

$f(-1) = -2(-2)(1)(-5) = -20$

4. $(x-2)(x+3)\overline{)\,p(x)\,}$ with quotient $q(x)$ above

$\ddots$

$\overline{ax+b}$

$P(x) = (x-2)(x+3)\,q(x) + ax+b$

$P(-3) = \qquad -3a+b = 17$

$P(2) = \qquad 2a+b = -3$

$-5a = 20 \Rightarrow a = -4$; then $b = 5$

[MML March 2006]

---

3. If you write $g(x) = a(x-2)(x-\frac{5}{2}) + b(x-\frac{3}{2})(x-\frac{5}{2}) + c(x-\frac{3}{2})(x-2)$

then $g(\frac{3}{2}) = a(-\frac{1}{2})(-1) = 2$, so $a = 4$. Similarly find $b = -12$, $c = 2$.

$g(3) = 4(1)(\frac{1}{2}) - 12(\frac{3}{2})(\frac{1}{2}) + 2(\frac{3}{2})(1) = -4$

Answers

Minnesota State High School Mathematics League

Team Event

2007-08 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

NO CALCULATORS IN THIS EVENT

11214 1. Express using base nine the integer which is written 54321 using base six.

$x_1 = 2$
$x_2 = 3$
$x_3 = 7$
2. Find positive integers x_1, x_2, and x_3, $x_1 < x_2 < x_3$ such that $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_1 x_2 x_3} = 1$.

3. Figure 3 shows a circle of radius 1 in which BD is tangent to the circle at C, and $AC \perp OB$. All six trigonometric functions of $\theta = \angle BOC$ can be expressed using a line segment shown on the figure. For example, $\sin\theta = AC$ and $\cos\theta = OA$. What line segments represent the other four functions? (Your answer must be a line segment shown in the figure; i.e. $\csc\theta = \frac{1}{AC}$ is not allowed.)

$\csc\theta =$ OD $\sec\theta =$ OB $\tan\theta =$ BC $\cot\theta =$ CD

one point for each correct answer

$\frac{3\sqrt{5}}{2}$ 4. The square $ABCD$ (Figure 4) has sides of length 4. E is the midpoint of CD, FG is the perpendicular bisector of AE, meeting it at H. Give the length of GH in exact form.

1089 5. K is a positive two digit number. When its digits are reversed to form the two digit number L, $\underset{L \neq K}{\wedge}$ then $K^2 - L^2$ is a perfect square. What is that perfect square?

$a = -2$
$b = 11$
$c = -17$
6. Each zero of $f(x) = ax^3 + bx^2 + cx + 7$ is one more than the reciprocal of a zero of $g(x) = x^3 + x^2 - 5x + 2$. Determine a, b, and c.

all three must be correct for credit

Team______________________

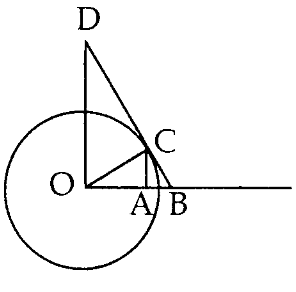


Figure 3

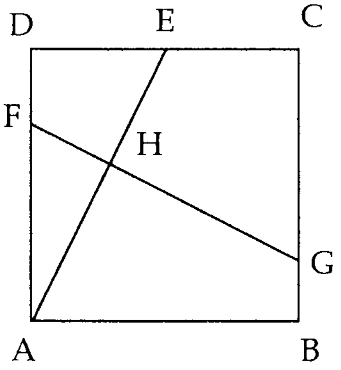


Figure 4

Meet 1 Solutions

1.
$$\begin{aligned} 1 &= 1 \\ 2\cdot 6 &= 12 \\ 3\cdot 6^2 &= 108 \\ 4\cdot 6^3 &= 864 \\ 5\cdot 6^4 &= 6480 \\ & \overline{7465} \end{aligned}$$

$$\begin{aligned} 7465 &= 1(6561)+1(729)+2(81)+1(9)+4 \\ &= 11214 \end{aligned}$$

2. $x_2x_3 + x_1x_3 + x_1x_2 + 1 = x_1x_2x_3$

$x_3(x_1x_2 - x_1 - x_2) = 1 + x_1x_2$

If we choose $x_3 = 1 + x_1x_2$, then we need to choose $x_1 < x_2$ such that $x_1x_2 - x_1 - x_2 = 1$. In 1A, #3, we found that $x_1 = 2$, $x_2 = 3$ works. Then $x_3 = 1 + 2\cdot 3 = 7$.

[adopted from BMK]

3. $\csc\theta = \frac{OD}{OC} = OD$

$\sec\theta = \frac{OB}{OC} = OB$

$\tan\theta = \frac{BC}{OC} = BC$

$\cot\theta = \frac{CD}{OC} = CD$

(Solutions use the fact that $\theta = \angle ODC$)

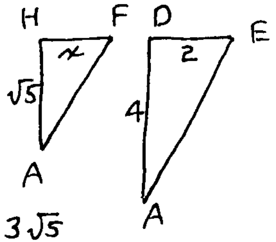
4. Drop perpendiculars from E and G to J and K on the opposite sides. Note that $\Delta FGK \cong \Delta AEJ$.

$\therefore\ FG = AE = \sqrt{16+4} = 2\sqrt{5}$

Next note $\Delta AFH \sim \Delta AED$

$\frac{x}{\sqrt{5}} = \frac{2}{4}$ so $x = \frac{\sqrt{5}}{2}$

$GH = FG - FH = 2\sqrt{5} - \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{2}$



5. Let $K = 10m + n$; then $L = 10n + m$

$$\begin{aligned} K^2 - L^2 &= 100m^2 + 20mn + n^2 \\ &\quad - (100n^2 + 20mn + m^2) \\ &= 99(m^2 - n^2) = 9\cdot 11\,(m+n)(m-n) \end{aligned}$$

Clearly either $m+n$ or $m-n$ must be 11, but $m-n$ won't work. $m+n = 11$. Also, since $m-n > 0$, $m > n$; and $m-n$ will have to be a perfect square. Consider the possibilities

m	n	m − n	
9	2	7	
8	3	5	
7	4	3	
6	5	1	← The only square

[MML March 2006]

$m = 6$; $n = 5$

The perfect square $= 9\cdot 11\cdot 11 = 1089$

6. Let the zeroes of $g(x)$ be r, s, and t. Then $rst = -2$, $rs + rt + st = -5$, $r + s + t = -1$ and the zeroes of $f(x)$ are

$\frac{1}{r} + 1$, $\frac{1}{s} + 1$, $\frac{1}{t} + 1$

$-\frac{7}{a} = \left(\frac{1}{r}+1\right)\left(\frac{1}{s}+1\right)\left(\frac{1}{t}+1\right)$

Multiply by rst,

$-\frac{7}{a}rst = (1+r)(1+s)(1+t)$

$-\frac{7}{a}rst = 1 + (r+s+t) + (rs+rt+st) + rst$

$-\frac{7}{a}(-2) = 1 + (-1) + (-5) + (-2) = -7$

$a = -2$

Proceed similarly from

$\frac{-b}{-2} = \frac{1}{r} + \frac{1}{s} + \frac{1}{t} + 3$; $b = 11$

$\frac{c}{-2} = \left(\frac{1}{r}+1\right)\left(\frac{1}{s}+1\right) + \left(\frac{1}{r}+1\right)\left(\frac{1}{t}+1\right) + \left(\frac{1}{s}+1\right)\left(\frac{1}{t}+1\right)$

$c = -17$