Solutions

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{239}{8}$ 1. Express as the quotient of two relatively prime integers

$$4^2 + 4^{\frac{3}{2}} + 4^1 + 4^0 + 4^{-\frac{1}{2}} + 4^{-1} + 4^{-\frac{3}{2}}$$

$\frac{a^2-1}{a^2+1}$ 2. Simplify $\dfrac{(a+a^{-1})^{-1}}{a(a^2-1)^{-1}}$, leaving an expression in which any exponents that appear are positive.

$-x$ 3. Given $f(x) = \dfrac{x - \frac{1}{x}}{x+1}$, find $\left[f\left(x^{-1}\right)\right]\left[f(x)\right]^{-1}$

4. Find all lattice points interior to the first quadrant, that is all points (m,n) where m and n are positive integers, that lie on the graph of

$$x^2 + y^2 + 2xy - 4x - 4y - 5 = 0$$

$(1,4), (2,3), (3,2), (4,1)$

1. $16+8+4+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 29 + \frac{7}{8} = \frac{239}{8}$

2. $\dfrac{(a+\frac{1}{a})^{-1}}{a(a^2-1)^{-1}} = \dfrac{(a-1)(a+1)}{a\left(a+\frac{1}{a}\right)} = \dfrac{a^2-1}{a^2+1}$

3. $f(x) = \dfrac{x^2-1}{x(x+1)} = \dfrac{x-1}{x}$

$f\left(\frac{1}{x}\right) = \dfrac{\frac{1}{x}-1}{\frac{1}{x}} = 1-x$

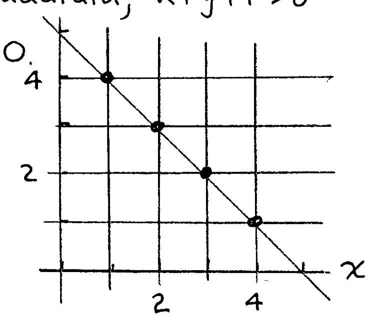
$f\left(\frac{1}{x}\right)\left[f(x)\right]^{-1} = (1-x)\dfrac{x}{x-1} = -x$

4. $(x+y)^2 - 4(x+y) - 5 = 0$

$[x+y-5][x+y+1] = 0$

In the first quadrant, $x+y+1>0$

$\therefore\ x+y-5=0.$



Lattice points:

$(1,4), (2,3), (3,2), (4,1)$

Solutions

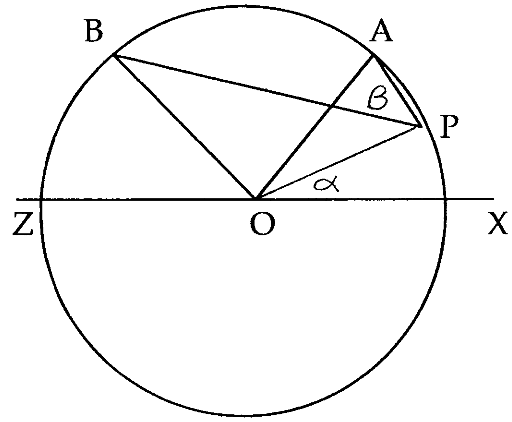
Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

All questions in this event refer to the figure at the right in which A and B are placed on a circle of radius 5 so that $\angle XOA = 45°$ *and* $\angle XOB = 135°$. *P is allowed to move along the circle so that* $\alpha = \angle XOP$ *varies from* 0° to 360°. *Let* $\beta = \angle APB$, *(so* β *is undefined when P coincides with either A or B).*



$\frac{15\pi}{4}$ 1. What is the length of the arc moving from A counterclockwise to Z?

$\frac{75\pi}{8}$ 2. What is the area of the region bounded by the arc described in Problem 1 and the two radii OA and OZ?

3. On the axes provided in Figure 3, graph β as a function of α as α varies from 0° to 360°. Use an open circle to show "gaps" in the graph at the points where β is undefined.

$\omega = \frac{\theta}{4}$ 4. In the given figure, suppose that with A still fixed, we now allow B to move on the circle so that $\theta = \angle BOA$ varies. Extend BP to M so that $AP = PM$. What is the measure of $\omega = \angle AMP$ in terms of θ?

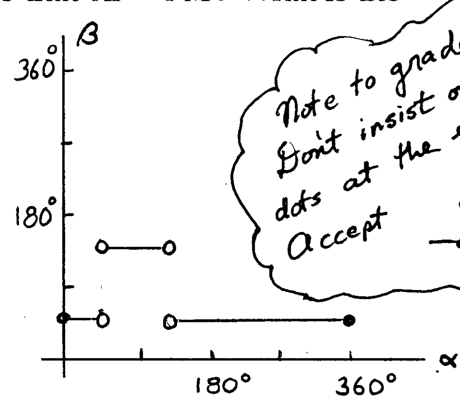


Figure 3

Note to graders: Don't insist on the solid dots at the end points. Accept

1. (radius)(central angle in radians) $= 5 \cdot \frac{3\pi}{4}$

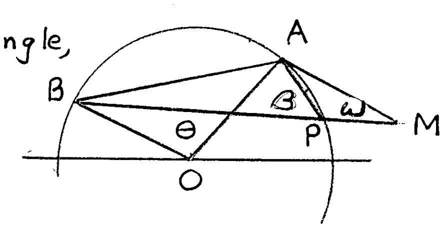
2. $\frac{r^2}{2}$ (central angle in radians) $= \frac{25}{2} \cdot \frac{3\pi}{4}$

3. For $0 \le \alpha < 45°$ $\beta = \frac{1}{2}\widehat{AB} = \frac{90}{2} = 45°$

For $45° < \alpha < 135°$ $\beta = \frac{1}{2}\widehat{BA} = \frac{270}{2} = 135°$

For $135° < \alpha \le 360$ $\beta = \frac{1}{2}\widehat{AB} = \frac{90}{2} = 45°$

4. Both β, an inscribed angle, and θ, a central angle, subtend $\widehat{AB}$, so $\beta = \frac{1}{2}\theta$.
ΔAMP is isosceles, so exterior angle $\beta = 2\omega$
$\therefore$ $2\omega = \frac{\theta}{2}$; $\omega = \frac{\theta}{4}$



Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS IN THIS EVENT

31 1. The sequence $\{a_n\}$ is defined recursively by

$$a_0 = 2,\ a_1 = 5, \text{ and } a_{n+2} = a_n + a_{n+1} \text{ for } n = 0,1,2,\ \ldots$$

Find a_5.

Accept 28a

28 2. The expansion of $\left(\frac{2}{a} + \frac{a^2}{4}\right)^8$ includes a term of the form ra where r is an integer. What is r?

(ii) 3. A function f is said to be *odd* if $f(x) = -f(-x)$ for all x. Indicate in the blank at the left (using i, ii, iii) which of the following functions are odd.

(i) $f(x) = -(x-2) + \frac{1}{3}(x-2)^3$

(ii) $f(x) = x\cos x$

(iii) $f(x) = x\sin x$

4. It is well known that the sum $1 + 2 + 3 + \ \ldots\ + n$ can be expressed in the closed form $\frac{n(n+1)}{2}$. Find a similar closed form for the sums

Give 1 point for each correct answer

$-\frac{n(n+1)}{2}$ (a) $1^2 - 2^2 + 3^2 - 4^2 + \ \ldots\ - n^2$ when n is even

$\frac{n(n+1)}{2}$ (b) $1^2 - 2^2 + 3^2 - 4^2 + \ \ldots\ + n^2$ when n is odd

1. 2, 5, 7, 12, 19, 31 = a_5

term of interest $= 8 \cdot 7 \frac{2^5}{2^6} a$

2. $\left(\frac{2}{a}\right)^8 + 8\left(\frac{2}{a}\right)^7\left(\frac{a^2}{4}\right) + \frac{8\cdot 7}{2}\left(\frac{2}{a}\right)^6\left(\frac{a^2}{4}\right)^2 + \underbrace{\frac{8\cdot 7\cdot 6}{3\cdot 2}\left(\frac{2}{a}\right)^5\left(\frac{a^2}{4}\right)^3}_{} + \ldots$

4. (a) n even

$[1^2 - 2^2] + [3^2 - 4^2] + \ldots + [(n-1)^2 - n^2]$

$= [(1-2)(1+2)] + [(3-4)(3+4)] + \ldots + [((n-1)-n)((n-1)+n)]$

$= -1\left\{(1+2) + (3+4) + \ldots + (n-1) + n\right\} = -\frac{(n+1)n}{2}$

(b) n odd

$1^2 - [2^2 - 3^2] - [4^2 - 5^2] - \ldots - [(n-1)^2 - n^2]$

$= 1 - [(2-3)(2+3)] - [(4-5)(4+5)] - \ldots - [((n-1)-n)((n-1)+n)]$

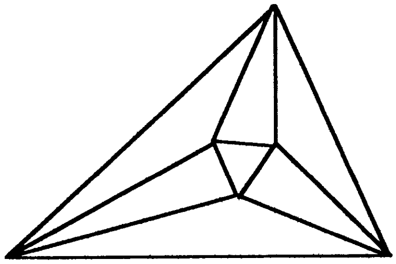
$= 1 + (2+3) + (4+5) + \ldots + (n-1) + n = \frac{n(n+1)}{2}$

3. (i) $-(-x-2) - \frac{1}{3}(x+2)^3 = (x+2) - \frac{1}{3}(x+2)^3 \neq -f(x)$

(ii) $-x\cos(-x) = -x\cos x = -f(x)$

(iii) $-x\sin(-x) = x\sin x \neq -f(x)$

Solutions



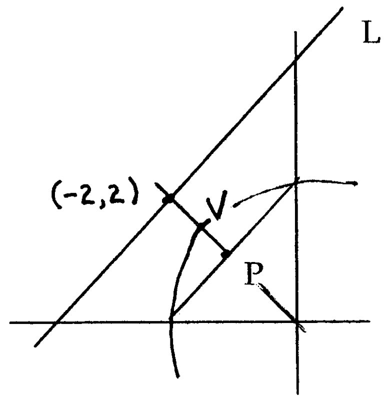
Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

All questions refer to the line L with equation $x - y + 4 = 0$ *and the point* $P(-1,1)$ *shown in the figure at the right.*



$\sqrt{2}$ 1. What is the distance from the point P to the line L?

$\left(-\frac{3}{2}, \frac{3}{2}\right)$ 2. Points equidistant from P and the line L lie on a parabola. Give the coordinates of the vertex of this parabola.

Accept 2

$(0,2)$ 3. Points equidistant from P and the line L lie on a parabola. Where does this parabola cross the y-axis?

4. Write in the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ the equation of the parabola described in problems 2 and 3.

$x^2 + 2xy + y^2 - 4x + 4y - 12 = 0$

1. The distance from P to $(-2,2)$ is $\sqrt{1^2+1^2} = \sqrt{2}$

2. The vertex is half way between P and L, at $\left(-\frac{3}{2}, \frac{3}{2}\right)$

3. The distance from P to L is $\sqrt{2}$. $(0,2)$ is also $\sqrt{2}$ from P ($VP = \frac{\sqrt{2}}{2}$; the line joining $(-2,0)$ and $(0,2)$ is the latus rectum; it has length $4\left(\frac{\sqrt{2}}{2}\right)$ and is centered at P)

4. distance from a point (x,y) on the curve to L is $\frac{x-y+4}{\sqrt{2}}$.
The distance to P is $\sqrt{(x+1)^2 + (y-1)^2}$

$$\frac{(x-y+4)^2}{2} = x^2 + 2x + 1 + y^2 - 2y + 1$$

Simplify and collect terms;

$$x^2 + 2xy + y^2 - 4x + 4y - 12 = 0$$

Answers

Minnesota State High School Mathematics League

Team Event

2006-07 Meet 4

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

k 1. How many lattice points are there that are interior to the first quadrant, that is all points (m,n) where m and n are positive integers, that lie on the graph of

$$x^2 + y^2 + 2xy - kx - ky - k - 1 = 0$$

2. Use inequalities to order from the smallest to the largest the numbers $5^{(5^5)}$, $(5!)!$, $(5^5)!$

$(5!)! < (5)^{(5^5)} < (5^5)!$

$\frac{n(n-1)}{2}$ 3. For $n > 4, f(x) = x^n - n(x-1) - 1$ can be written in the form $f(x) = (x-1)^2 g(x)$. Find, in terms of n, $g(1)$.

4. Let $x = \dfrac{1}{b-c}$, $y = \dfrac{1}{c-a}$, $z = \dfrac{1}{a-b}$. Then $x^2 + y^2 + z^2 = [f(x,y,z)]^2$ where $f(x,y,z)$ does not involve a,b, or c. Find $f(x,y,z)$ in simplified form.

$f(x,y,z) = x + y + z$

5. A function f is said to be *odd-like* for $a \le x \le b$ if $f(a+b-x) = -f(x)$. Indicate in the

(i), (ii) blank at the left (using i, ii, iii) which of the following functions are odd-like.

(i) $f(x) = -(x-2) + \frac{1}{3}(x-2)^3$ on (1,3)

(ii) $f(x) = x\cos x$ on $(0,\pi)$

(iii) $f(x) = x\sin x$ on $(0,\pi)$

6.355 6. The large circle in Figure 6 is tangent to both coordinate axes and the circle $x^2 + y^2 = 1$. The line joining the centers of the circles intersects the small circle at A and B, and CD is a diameter of the large circle that is perpendicular to the x-axis. Find the length of BD.

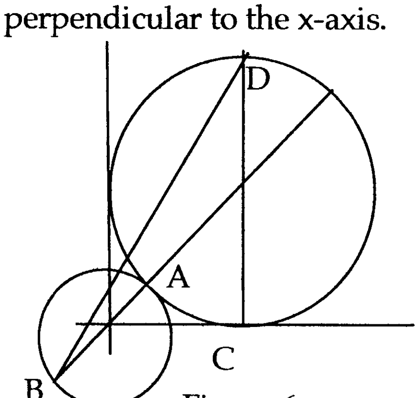


Figure 6

Team______________________________

Team Event 4 Solutions

1. $(x+y)^2 - k(x+y) - (k+1) = 0$

$[(x+y)+1][(x+y)-(k+1)] = 0$

Since $x+y+1 > 0$ in the 1st quad.,
$x+y = k+1$. Lattice point solns:
$(1, k), (2, k-1), \ldots, (k, 1)$.
There are k such solutions

2. $\underline{(5!)!} = 120! < (125)^{120} < 125^{1000}$

$\leq (5^3)^{1000} = 5^{3000} < 5^{3125} = \underline{5^{(5^5)}}$

$< 3125! = \underline{(5^5)!}$

3. Since $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + 1)$,

$f(x) = (x-1)[x^{n-1} + x^{n-2} + \ldots + 1 - n]$

Use synthetic division on the bracketed term;

$$\begin{array}{c|ccccc|c} & \multicolumn{5}{c|}{\overbrace{1 \quad 1 \quad 1 \ \ldots \quad 1}^{n-1}} & (1-n) \\ \hline 1 & 1 & 2 & 3 \ldots & & (n-1) & 0 \end{array}$$

$\therefore f(x) = (x-1)^2 \underbrace{[x^{n-2} + 2x^{n-3} + \ldots + (n-2)x + (n-1)]}_{g(x)}$

$g(1) = 1 + 2 + \ldots + (n-2) + (n-1) = \dfrac{n(n-1)}{2}$

4. Note that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = (b-c) + (c-a) + (a-b) = 0$

i.e. $\dfrac{yz + xz + xy}{xyz} = 0$, so the numerator is 0.

$(x+y+z)^2 = x^2 + y^2 + z^2 + 2\underbrace{(xy + xz + yz)}_{0}$

$\therefore x^2 + y^2 + z^2 = (x+y+z)^2$ so $f(x,y,z) = x+y+z$

5. (i) $f(4-x) = -(2-x) + \frac{1}{3}(2-x)^3 = (x-2) - \frac{1}{3}(x-2)^3 = -f(x)$

(ii) $f(\pi - x) = \cos(\pi - x) = -\cos x = -f(x)$

(iii) $f(\pi - x) = \sin(\pi - x) = \sin x \neq -f(x)$

6. Let R be the radius of the large circle. From $R^2 + R^2 = (R+1)^2$ we get $R = 1 + \sqrt{2}$, $R^2 = 3 + 2\sqrt{2}$

Draw $x = AD$. By the law of cosines

$x^2 = R^2 + R^2 - 2R^2 \cos 135° = R^2(2 + \sqrt{2})$

$\angle BAD = 180° - 2.5° = 157.5°$

Again appeal to the law of cosines.

$BD^2 = 4 + x^2 - 2(2)x \cos 157.5$

$x^2 = (3 + 2\sqrt{2})(2 + \sqrt{2}) = 10 + 7\sqrt{2}$

so $x = \sqrt{19.899} = 4.461$

$BD^2 = 40.385$

$BD = 6.355$

(In exact terms, $BD^2 = 22 + 13\sqrt{2}$)