Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

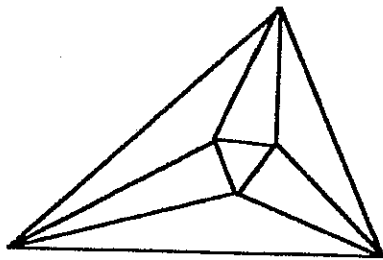_____ 1. Express as the quotient of two relatively prime integers
$$4^2 + 4^{\frac{3}{2}} + 4^1 + 4^0 + 4^{-\frac{1}{2}} + 4^{-1} + 4^{-\frac{3}{2}}$$

_____ 2. Simplify $\dfrac{\left(a + a^{-1}\right)^{-1}}{a\left(a^2 - 1\right)^{-1}}$, leaving an expression in which any exponents that appear are positive.

_____ 3. Given $f(x) = \dfrac{x - \dfrac{1}{x}}{x + 1}$, find $\left[f\left(x^{-1}\right)\right]\left[f(x)\right]^{-1}$

4. Find all lattice points interior to the first quadrant, that is all points (m, n) where m and n are positive integers, that lie on the graph of
$$x^2 + y^2 + 2xy - 4x - 4y - 5 = 0$$

_____

Name_____Team_____

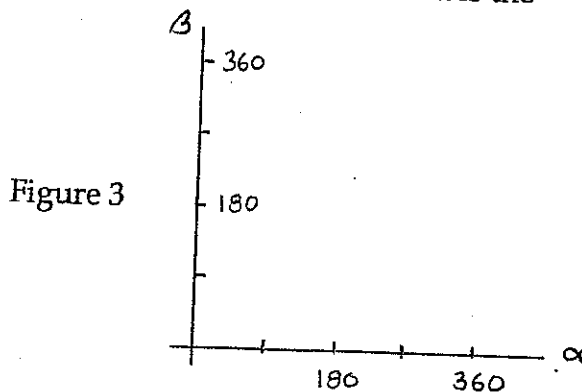# Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

All questions in this event refer to the figure at the right in which A and B are placed on a circle of radius 5 so that $\angle XOA = 45°$ and $\angle XOB = 135°$. P is allowed to move along the circle so that $\alpha = \angle XOP$ varies from 0° to 360°. Let $\beta = \angle APB$, (so β is undefined when P coincides with either A or B).
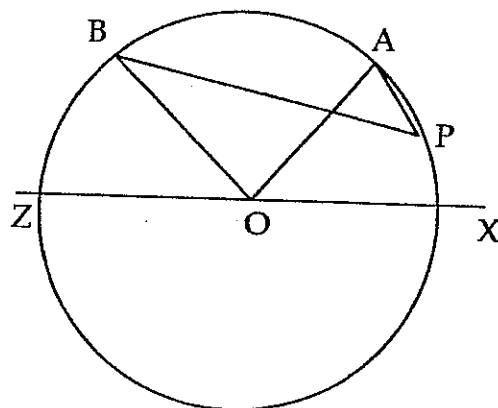
_____1. What is the length of the arc moving from A counterclockwise to Z?

_____2. What is the area of the region bounded by the arc described in Problem 1 and the two radii OA and OZ?

3. On the axes provided in Figure 3, graph β as a function of α as α varies from 0° to 360°. Use an open circle to show "gaps" in the graph at the points where β is undefined.

_____4. In the given figure, suppose that with A still fixed, we now allow B to move on the circle so that $\theta = \angle BOA$ varies. Extend BP to M so that $AP = PM$. What is the measure of $\omega = \angle AMP$ in terms of θ?

Figure 3

Name_____Team_____
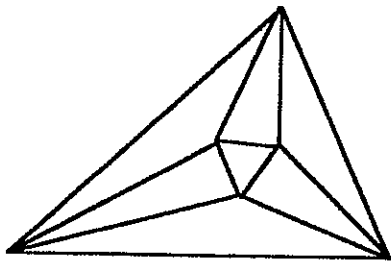
Minnesota State High School Mathematics League

Individual Event

2006-07 Event 4C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS IN THIS EVENT

_____1. The sequence $\{a_n\}$ is defined recursively by
$$a_0 = 2, \ a_1 = 5, \text{ and } a_{n+2} = a_n + a_{n+1} \text{ for } n = 0,1,2,\ldots$$
Find a_5.

_____2. The expansion of $\left(\dfrac{2}{a} + \dfrac{a^2}{4}\right)^8$ includes a term of the form ra where r is an integer.

What is r?

3. A function f is said to be *odd* if $f(x) = -f(-x)$ for all x. Indicate in the blank at the
_____ left (using i, ii, iii) which of the following functions are odd.

    (i) $f(x) = -(x-2) + \dfrac{1}{3}(x-2)^3$

    (ii) $f(x) = x\cos x$

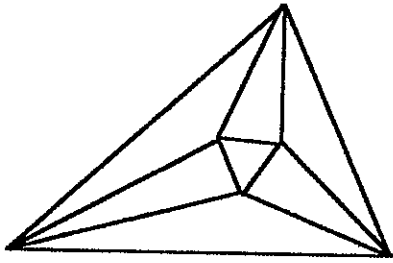   (iii) $f(x) = x\sin x$

4. It is well known that the sum $1 + 2 + 3 + \ldots + n$ can be expressed in the closed form
$\dfrac{n(n+1)}{2}$. Find a similar closed form for the sums

_____   (a) $1^2 - 2^2 + 3^2 - 4^2 + \ldots - n^2$ when n is even

_____   (b) $1^2 - 2^2 + 3^2 - 4^2 + \ldots + n^2$ when n is odd

Name_____Team_____

Minnesota State High School Mathematics League

## Individual Event

2006-07 Event 4D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

All questions refer to the line L with equation
$x - y + 4 = 0$ and the point $P(-1,1)$ shown in
the figure at the right.
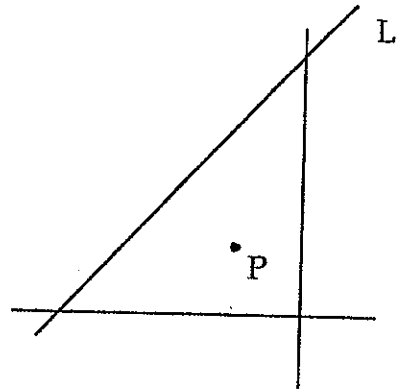
_____ 1. What is the distance from the point P to the line L?

2. Points equidistant from P and the line L lie on a parabola. Give the coordinates of
_____   the vertex of this parabola.

3. Points equidistant from P and the line L lie on a parabola. Where does this
_____   parabola cross the y-axis?

4. Write in the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ the equation of the parabola
   described in problems 2 and 3.

_____

Name_____Team_____
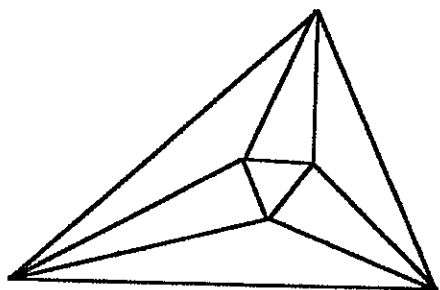
Minnesota State High School Mathematics League

Team Event

2006-07 Meet 4

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

_____ 1. How many lattice points are there that are interior to the first quadrant, that is all points (m,n) where m and n are positive integers, that lie on the graph of
$$x^2 + y^2 + 2xy - kx - ky - k - 1 = 0$$

2. Use inequalities to order from the smallest to the largest the numbers
_____ $5^{(5^5)}$, $(5!)!$, $(5^5)!$

3. For $n > 4$, $f(x) = x^n - n(x-1) - 1$ can be written in the form $f(x) = (x-1)^2 g(x)$. Find,
_____ in terms of n, $g(1)$.

4. Let $x = \dfrac{1}{b-c}$, $y = \dfrac{1}{c-a}$, $z = \dfrac{1}{a-b}$. Then $x^2 + y^2 + z^2 = [f(x,y,z)]^2$ where $f(x,y,z)$
_____ does not involve a, b, or c. Find $f(x,y,z)$ in simplified form.

5. A function f is said to be *odd-like* for $a \le x \le b$ if $f(a+b-x) = -f(x)$. Indicate in the
_____ blank at the left (using i, ii, iii) which of the following functions are odd-like.
   (i) $f(x) = -(x-2) + \dfrac{1}{3}(x-2)^3$ on $(1,3)$
   (ii) $f(x) = x\cos x$ on $(0,\pi)$
   (iii) $f(x) = x\sin x$ on $(0,\pi)$

_____ 6. The large circle in Figure 6 is tangent to both coordinate axes and the circle $x^2 + y^2 = 1$. The line joining the centers of the circles intersects the small circle at A and B, and CD is a diameter of the large circle that is perpendicular to the x-axis. Find the length of BD.
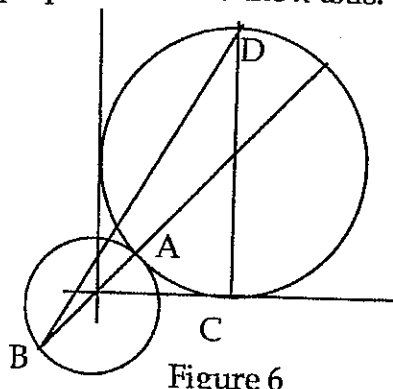
Team_____

Figure 6