SOLUTIONS

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 2A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{5}{2}$ 1. Find x if $4(x-7)=3(2x-11)$.

$\frac{8}{5a}$ 2. Find x if $3(ax-3)-2(ax-2)=5(ax-2)+ax-3$.

$\frac{16}{3}$ 3. There is an interval on the x-axis for which $5(x-2)\ge 2(x-7)$ and $11-4x\ge -5$. How long is this interval?

15 4. I notice when riding in a light rail car that there is a distinct clicking sound each time the wheels come to a new section of track. If I begin timing myself just after I hear a click, for how many seconds should I count so that the number of clicks counted will equal the speed of the train in miles per hour? (There are 5280 feet in a mile, and the sections of rail are 22 feet long.)

1. $4x-28=6x-33$

$5=2x$

$x=\frac{5}{2}$

2. $2(ax-3)=7(ax-2)$

$2ax-6=7ax-14$

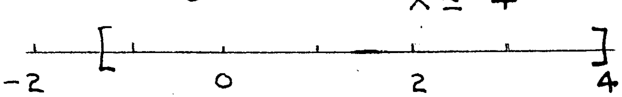
$8=5ax$

$x=\frac{8}{5a}$

3. $5x-10\ge 2x-14$ $\quad$ $11-4x\ge -5$

$3x \ge -4$ $\quad$ $-4x\ge -16$

$x \ge -\frac{4}{3}$ $\quad$ $x\le 4$



$4-\left(-\frac{4}{3}\right)=\frac{12+4}{3}=\frac{16}{3}$

4. A train traveling n mph travels

$\frac{5280}{3600}n=\frac{22n}{15}$ ft/sec.

If the rails are 22 feet long, we will count $\frac{n}{15}$ clicks/sec.

We want $\frac{n}{15}$ (seconds counting) $=n$.

$\therefore$ We should count for 15 seconds.

SOLUTIONS

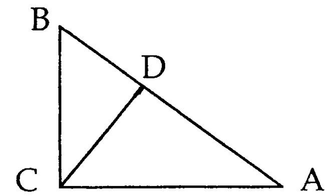
Minnesota State High School Mathematics League

Individual Event

2006-07 Event 2B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

All questions refer to the Figure at the right which shows a right triangle ABC with legs $BC = 3$, $CA = 4$. A line segment has been drawn from C to D so that $CD \perp AB$.



$\frac{12}{5}$ 1. What is the length of CD?

$\frac{16}{3}$ 2. Extend CD until it intersects at E a line erected at A to be perpendicular to AC. Find the length of AE.

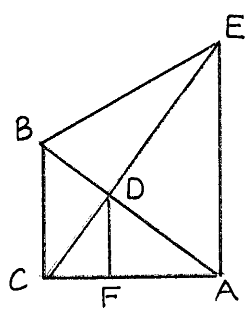
$\frac{36}{25}$ 3. From D, drop a line perpendicular to AC, meeting AC at F. What is the length of CF?

$\frac{\sqrt{193}}{3} = 4.631$ 4. What is the length of BE?

1. $\frac{CD}{CB} = \frac{CA}{AB}$

$\frac{CD}{3} = \frac{4}{5}$

$CD = \frac{12}{5}$



2. $\triangle ECA \sim \triangle ABC$

$\frac{AE}{AC} = \frac{CA}{CB} = \frac{4}{3}$

$AE = \frac{4}{3}(4) = \frac{16}{3}$

3. $\triangle DCF \sim \triangle ABC$

$\frac{CF}{CD} = \frac{BC}{BA} = \frac{3}{5}$

$CF = \frac{3}{5} \cdot \frac{12}{5} = \frac{36}{25}$

4. $BE^2 = BD^2 + DE^2$

$\triangle CBD \sim \triangle ABC$

$\frac{BD}{BC} = \frac{BC}{BA} = \frac{3}{5}$

$BD = 3\left(\frac{3}{5}\right) = \frac{9}{5}$

$DA = 5 - BD = 5 - \frac{9}{5} = \frac{16}{5}$

$\triangle EAD \sim \triangle ABC$

$\frac{DE}{DA} = \frac{4}{3}$

$DE = \frac{4}{3} \cdot \frac{16}{5} = \frac{64}{15}$

$\therefore BE^2 = \left(\frac{9}{5}\right)^2 + \left(\frac{64}{15}\right)^2$

$= \frac{9 \cdot 81}{9 \cdot 25} + \frac{(64)^2}{9 \cdot 25}$

$= \frac{25(193)}{9 \cdot 25} = \frac{193}{9}$

$BE = \frac{\sqrt{193}}{3} = 4.631$

Solutions

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 2C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1 ____ 1. Figure 1 shows a right triangle ABC with legs $BC = 3$, $CA = 4$. Find $\sin(\angle A + \angle B)$.

$\frac{4}{3}$ ____ 2. In ΔABC shown in Figure 1, let the line bisecting $\angle A$ intersect BC at D. What is the length of CD?

3. Derive a formula for $\tan 3\theta$ that involves $\tan\theta$ and no other trigonometric functions.

$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ ____

4. In a certain ΔABC, $\cos A\cos B + \sin A\sin B\sin C = 1$. Find the measure in degrees of $\angle A$, $\angle B$, and $\angle C$. [BKM - 225]

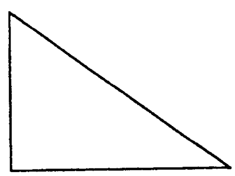
$\angle A = 45°$ ____

$\angle$ = 45° ____

$\angle C = 90°$ ____

no partial credit for #4

B



C A

Figure 1

1. Since $A + B = 90°$,

$\sin(A+B) = 1$

2. $\dfrac{CD}{4} = \tan\dfrac{\angle A}{2}$

Let $\alpha = \angle A$

$\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} = \sqrt{\dfrac{1-4/5}{1+4/5}} = \dfrac{1}{3}$

$CD = 4\left(\frac{1}{3}\right)$

3. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\tan(\theta + 2\theta) = \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}$

$\tan 3\theta = \dfrac{\tan\theta + \dfrac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\dfrac{2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Graders: Complex fract are prohibited by league rule.

4. Let the term $\cos A\cos B$ remind you that $\cos(A-B) = \cos A\cos B + \sin A\sin B$. The given equation enables us to write this as

$$\cos(A-B) = [1 - \sin A\sin B\sin C] + \sin A\sin B = 1 + \sin A\sin B(1-\sin C)$$

Now $\cos(A-B) \le 1$ while $1 + \sin A\sin B(1-\sin C) \ge 1$. It follows that

$$1 + \sin A\sin B(1-\sin C) = 1$$

$$\sin A\sin B(1-\sin C) = 0$$

Since A and B are angles in a Δ, $\sin A > 0$ and $\sin B > 0$; $1 - \sin C = 0$.

$C = 90$, $\cos(A-B) = 1 \Rightarrow A - B = 0$; $A = B = 45°$

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 2D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Write an equation of the line tangent to the graph of the circle $x^2 + y^2 = 5$ at (2,1).

$y = -2x + 5$

2. The graph of $xy = 1$ is symmetric about the line L having equation $y = x$. Write the equation of the line tangent to the graph of $xy = 1$ at the point where it crossed the line L in the third quadrant.

$y = -x - 2$

3. Write an equation of the circle of radius $\sqrt{20}$ that is tangent to the graph of the line $x + 2y = 5$ at (3,1) and lies entirely in the first quadrant.

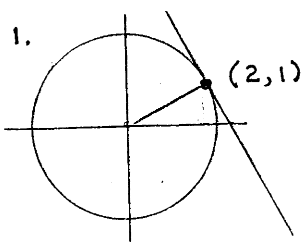
$(x-5)^2 + (y-5)^2 = 20$ OR $x^2 - 10x + y^2 - 10y + 30 = 0$

Two acceptable forms.

4. A $30° - 60° - 90°$ ΔABC with hypotenuse 3 and an isosceles right ΔEBD with hypotenuse $\sqrt{6}$ are positioned as in Figure 4 with their hypotenuses intersecting at F. Give in exact form the length of the line dropped from F perpendicular to AB at G.

See the solution

1.

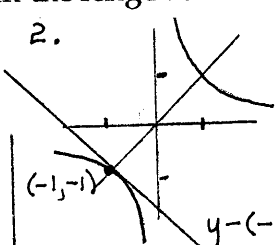


slope $= -2$

$y - 1 = -2(x-2)$

or $y = -2x + 5$

2.



$y - (-1) = -1(x+1)$

or $y = -x - 2$

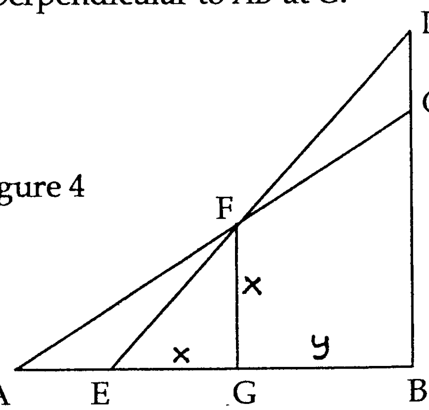
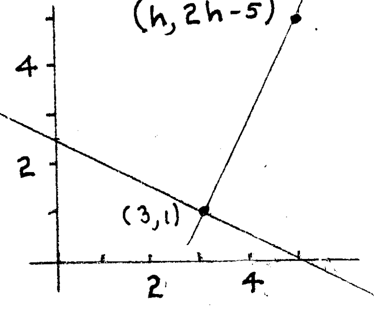


Figure 4

3. The center, on the line $y - 1 = 2(x-3)$, is of the form, $(h, 2h-5)$. It's distance from (3,1) is $\sqrt{20} = \sqrt{(h-3)^2 + (2h-6)^2} = (h-3)\sqrt{5}$ so $h = 5$,

$(x-5)^2 + (y-5)^2 = 20$



Tangent line has slope $= -\frac{1}{2}$

As in Fig 4, set $x = FG = EG$, $y = GB$, and $z = EB = x + y$.

Then $x = z - y = \sqrt{3} - y$ because $z^2 + z^2 = 6$.

Now $BC = \frac{1}{2} AC = \frac{3}{2}$ so $AB = \frac{3\sqrt{3}}{2}$

$y = AB - AG = \frac{3\sqrt{3}}{2} - x\sqrt{3}$

$\therefore\ x = \sqrt{3} - \left(\frac{3\sqrt{3}}{2} - x\sqrt{3}\right) = \frac{-\sqrt{3}}{2} + x\sqrt{3}$

$x = \frac{-\sqrt{3}}{2(1-\sqrt{3})} = \frac{3+\sqrt{3}}{4}$ (accept either form)

Answers

Minnesota State High School Mathematics League

Team Event

2006-07 Meet 2

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

$\left(\frac{36}{25}, \frac{48}{25}\right)$ 1. In Event 2B, a 3-4-5 right ΔABC was described in which a line was drawn from C to D on the hypotenuse making $CD \perp AB$. If coordinate axes are drawn with the origin at C and we have $A(4,0)$ and $B(0,3)$ (Figure 1), what will be the coordinates of D?

2. Write in the form $Ax + By + C = 0$ the equation of a line drawn in Figure 1 that is parallel to AB and exactly 3 units below point C (i.e. the origin).

$3x + 4y = 15$

$(1,1)$ 3. In Figure 1, let the bisectors of $\angle CAB$ and $\angle ABC$ intersect at E. Find the coordinates of E.

4. Write an equation of the circle of radius $\sqrt{20}$ that is tangent in the first quadrant to the graph of the line $x + 2y = 5$ and passes through $(-5,-4)$.

$(x+1)^2 + (y+2)^2 = 20$ OR $x^2 + 2x + y^2 + 4y = 15$

12 5. The y-intercept of a line passing through $\left(\frac{5}{2}, 3\right)$ is $\frac{1}{3}$ of the x-intercept. What is the smallest integer value of x for which the graph of this line is in the fourth quadrant?

3.907 OR $\sqrt{\frac{25+12\sqrt{3}}{3}}$ 6. Napoleon's Theorem says that if equilateral triangles are constructed on the three sides of an arbitrary triangle, then the centers of these three triangles will themselves be vertices of an equilateral triangle. Beginning with ΔABC in Figure 1 and constructing the three equilateral triangles exterior to ΔABC, what will the length of the sides of the equilateral triangle guaranteed by Napoleon's Theorem?

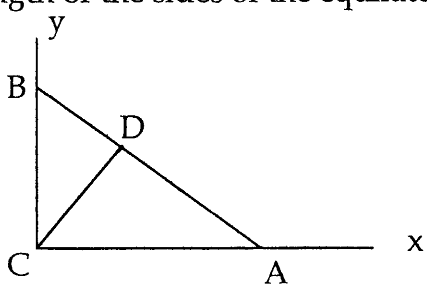


Figure 1

TEAM EVENT 2 SOLUTIONS

1. Line AB: $3x + 4y = 12$

Line CD: $-4x + 3y = 0$

Solve to get $x = \frac{36}{25}$, $y = \frac{48}{25}$

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2. Line AB in normal form:

$$\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$$

A line ∥ to AB, 3 units below the origin, in normal form:

$$\frac{3}{5}x + \frac{4}{5}y = -3, \text{ or}$$

$$3x + 4y = -15$$

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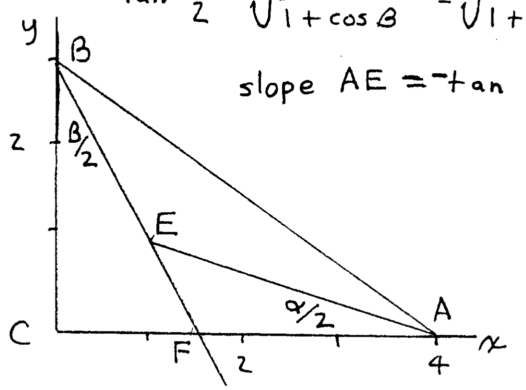
3. Let $\alpha = \angle CAB$, $\beta = \angle ABC$.

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{1-4/5}{1+4/5}} = \frac{1}{3}$$

$$\tan\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{1+\cos\beta}} = \sqrt{\frac{1-3/5}{1+3/5}} = \frac{1}{2}$$



slope AE $= -\tan\frac{\alpha}{2} = -\frac{1}{3}$

Let the extension of BE intersect CA at F, forming $\angle AFE$

$$\tan(\angle AFE) = -\tan\angle CFE = -\cot\frac{\beta}{2}$$

$$= -\frac{1}{\tan\beta/2} = -2$$

slope of FE $= -2$

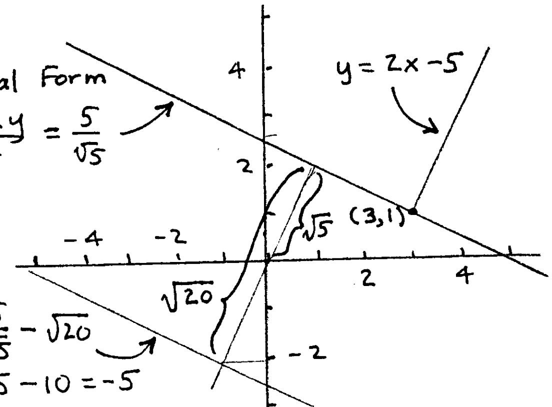
AE: $y = -\frac{1}{3}(x-4)$

Line FE: $y - 3 = -2x$

Solve $\begin{cases} x + 3y = 4 \\ 2x + y = 3 \end{cases}$ $E(1,1)$

4.

Normal Form $\frac{x+2y}{\sqrt{5}} = \frac{5}{\sqrt{5}}$



$\frac{x+2y}{\sqrt{5}} = \frac{5}{\sqrt{5}} - \sqrt{20}$

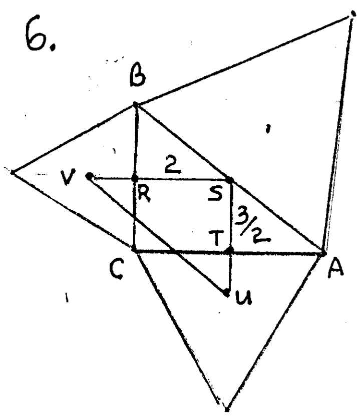
$x + 2y = 5 - 10 = -5$

The center is of the form $(-2k-5, k)$ and its distance from $(-5,-4)$ is $\sqrt{20}$

$$[-2k-5-(-5)]^2 + [k-(-4)]^2 = 20$$

Solving gives $k = \frac{2}{5}$ or $k = -2$. Possible centers are $\left(-\frac{24}{5}, \frac{2}{5}\right)$ or $(-1,-2)$. Only $(-1,-2)$ gives a point of tangency in 1^{st} quad.

6.



$$VR = \frac{1}{3}\left[\frac{3}{2}\sqrt{3}\right] = \frac{\sqrt{3}}{2}$$

$$RS = \frac{1}{2}CA = 2$$

$$VS = VR + RS = \frac{\sqrt{3}+4}{2}$$

$$UT = \frac{1}{3}\left[2\sqrt{3}\right] = \frac{2\sqrt{3}}{3}$$

$$TS = \frac{1}{2}CB = \frac{3}{2}$$

$$US = UT + TS = \frac{4\sqrt{3}+9}{6}$$

$$VU^2 = VS^2 + US^2$$

$$= \left(\frac{\sqrt{3}+4}{2}\right)^2 + \left(\frac{4\sqrt{3}+9}{6}\right)^2$$

$$= \frac{3+8\sqrt{3}+16}{4} + \frac{48+72\sqrt{3}+81}{36}$$

$$= \frac{57+24\sqrt{3}}{12} + \frac{43+24\sqrt{3}}{12}$$

$$VU^2 = \frac{100+48\sqrt{3}}{12} = \frac{25+12\sqrt{3}}{3}$$

$$VU = 3.907$$

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5. Let the y-intercept be b; then the x-intercept is $3b$. Substitute $x = \frac{5}{2}$, $y = 3$ into $\frac{x}{3b} + \frac{y}{b} = 1$ to get $b = \frac{23}{6}$

The smallest integer greater than $3b = \frac{23}{2}$ is 12.