SOLUTIONS

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 1A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

3640 1. Express as a single integer the least common multiple of the set {52, 56, 70}.

$\frac{8}{21}$ 2. Express $\dfrac{\frac{5}{63}+\frac{3}{35}}{\frac{7}{45}+\frac{5}{18}}$ as the quotient of two relatively prime integers.

31 3. Find the smallest positive integer k such that $\frac{7}{39}+\frac{k}{117}=\left(\frac{a}{b}\right)^2$ where a and b are relatively prime positive integers.

d= 7
r= −12
s= 5
4. If d is the greatest common divisor of 399 and 959, then it is possible to find integers r and s so that $d=399r+959s$. Find d, r, and s.

1.
$52 = 2\cdot2\cdot13$
$56 = 2\cdot2\cdot2\cdot7$
$70 = 2\cdot5\cdot7$
$\therefore\ lcm = 2^3\cdot5\cdot7\cdot13$
$= 3640$

2.
$$\frac{\frac{5}{3^2\cdot7}+\frac{3}{5\cdot7}}{\frac{7}{3^2\cdot5}+\frac{5}{2\cdot3^2}}\cdot\frac{2\cdot3^2\cdot5\cdot7}{2\cdot3^2\cdot5\cdot7}=\frac{50+54}{98+175}=\frac{104}{273}=\frac{8}{21}$$

3.
$$\frac{7}{3\cdot13}+\frac{k}{3\cdot3\cdot13}=\frac{21+k}{9\cdot13}$$
Choose k so that $\frac{21+k}{13}=a^2$

k	$\frac{21+k}{13}$	
5	2	$\neq a^2$
18	3	$\neq a^2$
31	4	$= 2^2$

Choose $k = 31$.

4.
$(959) = 2(399) + (161)$
$(399) = 2(161) + (77)$
$(161) = 2(77) + (7)$
$(77) = 11(7)$
$d = 7$

$7 = [(959) - 2(399)] - 2[(399) - 2(161)]$
$= (959) - 4(399) + 4[(959) - 2(399)]$
$= 5(959) - 12(399)$
$r = -12, \quad s = 5$

Graders note:
$r = -12 + 137n$
$s = 5 - 57n$
will work for any n

SOLUTIONS

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 1B

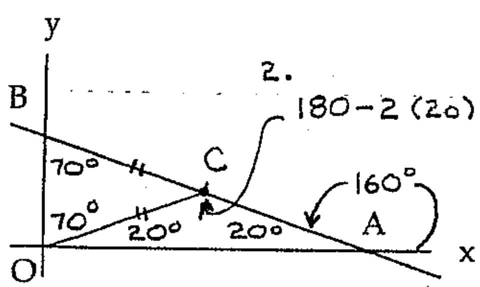
The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

160° 1. A straight line intersects the x-axis at A, and the y-axis at B as shown in Figure 1, making $\angle ABO = 70^\circ$. What is the measure of the supplement of $\angle OAB$?

140° 2. Referring to Figure 1 and the information given for Problem 1, suppose C is chosen between A and B so that $OC = BC$. What will be the measure of $\angle OCA$?

2α 3. In right ΔABC, let D be the mid-point of the hypotenuse BC, and let α be the measure of $\angle BCA$. In terms of α, what is the measure of $\angle ADB$?

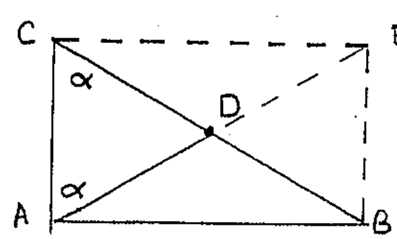
85° 4. Isosceles ΔABC has its vertex at $\angle A = 30^\circ$ (Figure 3). A trisector of $\angle A$ and a trisector of $\angle B$ meet at R. A trisector of $\angle B$ and a trisector of $\angle C$ meet at S. What is the measure of $\angle BSR$?



2. $180-2(20)=140^\circ$

Figure 1

3.



From comments below, ΔABD is isosceles with base angles of $(90-\alpha)$

$\therefore \angle ADB = 180-2(90-\alpha)$

$\angle ADB = 2\alpha$

Complete rectangle ABCE. Its diagonals intersect at D, making $AD = CD$, so $\angle CAD = \alpha$

This also follows directly from a theorem about exterior angles

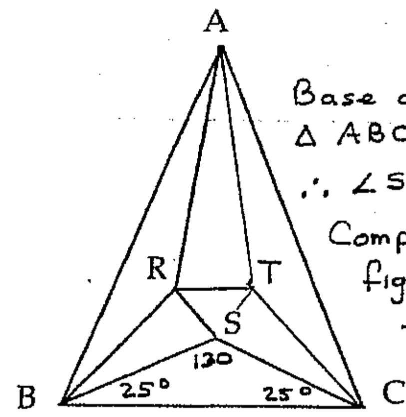


Figure 4

Base angles of ΔABC are $\frac{180-30}{2} = 75°$

$\therefore \angle SBC = 25°$

Complete the figure by drawing the other trisectors of $\angle A$ and $\angle C$.

By Morley's Thm, ΔRST is equilateral, making it easy to show $\Delta BSR \cong \Delta CST$. Let $\alpha = \angle BSR$. Summing angles around S,

$130+\alpha+60+\alpha = 360$

$2\alpha = 170$; $\alpha = 85°$

Solutions

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{13}{12}$ ← Accept $\frac{13}{12}\pi$

1. An angle of 195° has a radian measure of $r\pi$ where r is a rational number. What is r?

$195 = 180 + 15 = \pi + \frac{1}{3}\frac{\pi}{4} = \frac{13}{12}\pi$

$\frac{\sqrt{21}}{5}$ 2. The smallest acute angle of a right triangle has a sine of 0.4. In exact terms (not a decimal), what is the sine of the largest acute angle?

3. Given that $\cos\alpha > \cos\beta > \frac{1}{\sqrt{2}}$, consider the following three statements.

(a) $\alpha < \beta$ (b) $\alpha > \beta$ (c) $|\alpha| < |\beta|$

Answer each of the two questions below with as many of a, b, and c as seem correct, or answer *none*.

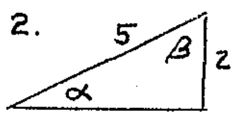
none Which statements must be true?

none Which statements must be false?

Give 1 point for each correct answer

$2\sqrt{6}$ 4. Figure 4 shows ΔABC with $AC = 4$, $BC = 2$, and a perpendicular dropped from C to D on AB so that $AD = 3DB$. What is the length of AB?

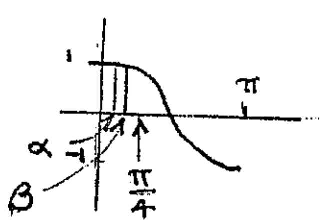
2.



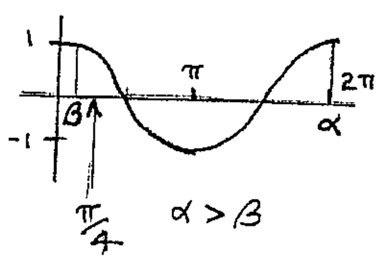
Let $\sin\alpha = .4 = \frac{2}{5}$

Then the side opposite β is $\sqrt{21}$; $\sin\beta = \frac{\sqrt{21}}{5}$

3. In both pictures below, $\cos\alpha > \cos\beta > \frac{1}{\sqrt{2}}$



$\alpha < \beta$

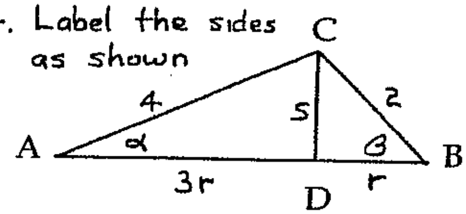


$\alpha > \beta$

$|\alpha| > |\beta|$

None of the statements must be true or false

4. Label the sides as shown



$\sin^2\alpha + \cos^2\alpha = \frac{s^2}{16} + \frac{9r^2}{16} = 1$

$\sin^2\beta + \cos^2\beta = \frac{s^2}{4} + \frac{r^2}{4} = 1$

Multiply the 2nd eq by $-\frac{1}{4}$ and add

$\frac{9r^2 - r^2}{16} = \frac{3}{4}$ so $8r^2 = 12$

$r^2 = \frac{12}{8}$ so $r = \frac{\sqrt{6}}{2}$

$AB = 4r = 2\sqrt{6}$

SOLUTIONS

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

No Calculators in this Event

$3/2$, $2/3$ 1. Find the roots of $6x^2-13x+6=0$. Both correct answers required for 1 point

2. Write in descending powers of x the equation of a minimal degree polynomial with integer coefficients having $1-i$ and $\frac{1}{2}$ as roots.

$2x^3-5x^2+6x-2$ Give credit if they include $=0$

$y=-2$ 3. Write the equation of the horizontal line that will be tangent to the graph of $x^2-6x+2y+13=0$.

4 4. If p, q, and r are the roots of $x^3-x^2+x-2=0$, what is the value of $p^3+q^3+r^3$?

1. $(2x-3)(3x-2)=0$

$x=\frac{3}{2}$; $x=\frac{2}{3}$

2. $(1-i)$ and $(1+i)$ must be roots

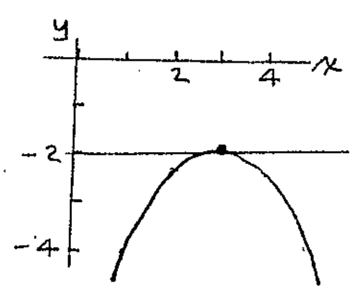
$[x-(1-i)][x-(1+i)]=(x-1)^2-i^2$

$=x^2-2x+1+1 = x^2-2x+2.$

Now multiply $(x^2-2x+2)(2x-1)$

$=2x^3-5x^2+6x-2$

3. The graph of $y=-\frac{1}{2}(x^2-6x+13)$ is at the right. To see that the maximum of y is 2, write



$y=-\frac{1}{2}\left[(x-3)^2+4\right]$

$=-2-\frac{1}{2}(x-3)^2$

Taken from [AHSME, 1975]

4. $x^3-x^2+x-2=(x-p)(x-q)(x-r)$

$=x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr.$

$\therefore$ $p+q+r=1$

$pq+pr+qr=1$

$pqr=2$

Since p, q, and r are roots,

$p^3-p^2+p-2=0$

$q^3-q^2+q-2=0$

$r^3-r^2+r-2=0$

Adding,

(*) $p^3+q^3+r^3-(p^2+q^2+r^2)+(p+q+r)=6$

Since

$(p+q+r)^2=(p^2+q^2+r^2)+2(pq+pr+qr),$

$1 = p^2+q^2+r^2+2(1)$

Substitute in (*);

$p^3+q^3+r^3-(-1)+1=6$

$p^3+q^3+r^3=4$

Answers

Minnesota State High School Mathematics League

Team Event

2006-07 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

11 1. Two mirrors AB and AC are set at 8° as in Figure 1. A light source is reflected at R_1 where the angle of incidence equals the angle of reflection as indicated. It is then reflected in a similar fashion at R_2, R_3, etc., until, on the nth reflection, it strikes one of the mirrors at a right angle, and then it retraces its path back to C. What is the largest possible value of n?

3,200,000 2. A newspaper reports that a wall to be built between two warring factions in a city will cost \$2 million per kilometer. Using the fact that a kilometer is .62 miles, how much, to the nearest \$100,000, will the wall cost per mile?

$\frac{\sqrt{3}-1}{3}$ 3. The right ΔADE in Figure 3 has a side of length 1 opposite the 30° angle at A. From E, lines are drawn to B and C on AD making $\angle EBD = 45°$ and $\angle ECD = 60°$. If a line perpendicular to AD erected at C intersects BE at H, how long (exact form) is HF?

104.5 4. Figure 4 shows ΔABC with $AC = 4$, $BC = 2$, and a perpendicular dropped from C to D on AB so that $AD = 3DB$. To the nearest tenth of a degree, what is the measure of $\angle ACB$?

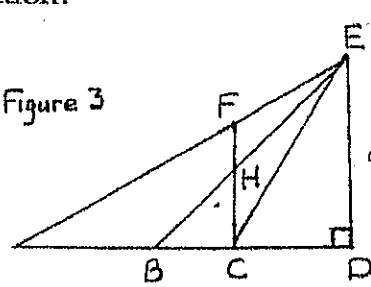
$\frac{25}{4}$ 5. The graph of $y = \dfrac{x^2 - x - 4}{2(x-3)}$ has a vertical asymptote and an asymptote skew to the x-axis. Find the area enclosed by the two asymptotes and the x-axis.

6. For what choices of k will the graphs of $y = k$ and $y = 2x^3 - 7x^2 - 12x + 6$ have exactly two distinct points of intersection?

$\frac{278}{27}$, -39

Give 2 points for each correct answer

Figure 3



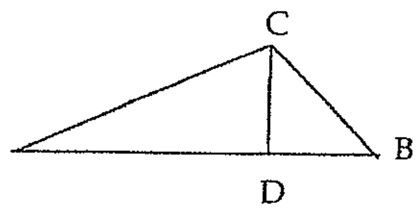
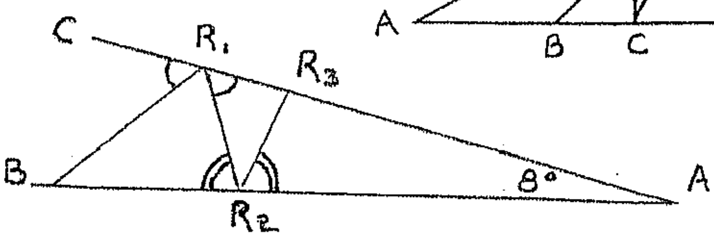


Figure 4

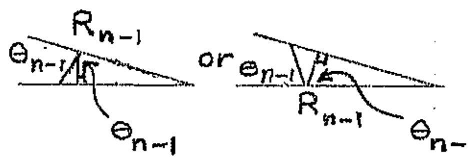
Figure 1



Team Event 1 Solutions

Taken from [AHSME, 1981, No. 20]

1. See Figure 1 on the answer sheet for notation; $\theta = \angle ABR_1$.
Let θ_1 be the equal angles at R_1
" θ_2 " " " " " R_2
$\vdots$
" θ_n " " " " " R_n

Either 

In either case, $\theta_{n-1} \pm 8 = 90$
Now, using the fact that exterior angles are sums of remote interior angles, we have

$\theta_1 = \theta + 8$
$\theta_2 = \theta_1 + 8 = \theta + 2(8)$
$\theta_3 = \theta_2 + 8 = \theta + 3(8)$
$\vdots$
$\theta_{n-1} = \theta_{n-2} + 8 = \theta + (n-1)8$

$\therefore$ $82 = \theta + (n-1)8$ Since $\theta > 0$, the largest choice of $n = 11$.

2. $\dfrac{\text{cost/mile}}{2{,}000{,}000} = \dfrac{1}{.62} = 1.613$

cost/mile $\approx 3{,}226{,}000 \approx$ * 3.2 million

3. We use the notation of Figure 3

$HC = BC = BD - CD = ED - CD = 1 - \frac{1}{\sqrt{3}} = \frac{3-\sqrt{3}}{3}$

$FC\sqrt{3} = AC = \sqrt{3} - \frac{1}{\sqrt{3}}$ so $FC = 1 - \frac{1}{3} = \frac{2}{3}$

$HF = \frac{2}{3} - \frac{3-\sqrt{3}}{3} = \frac{\sqrt{3}-1}{3}$

4. As in Problem 4 of Event C, show that $DB = \frac{\sqrt{6}}{2}$. Then $AD = \frac{3\sqrt{6}}{2}$

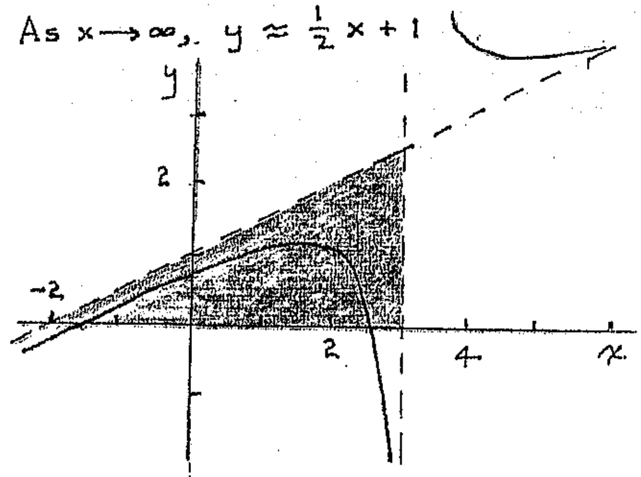
$\cos(\angle CAD) = \dfrac{3\sqrt{6}/2}{4} = \dfrac{3\sqrt{6}}{8}$ so $\angle CAD = 23.3°$

$\cos(\angle CBD) = \dfrac{\sqrt{6}/2}{2} = \dfrac{\sqrt{6}}{4}$ so $\angle CBD = 52.2°$

$\therefore \angle ACB = 180° - (23.3 + 52.2) = 104.5$

5. By long division, $y = \frac{x}{2} + 1 + \frac{2}{2x-6}$

As $x \to \infty$, $y \approx \frac{1}{2}x + 1$



The shaded area bounded by $y = \frac{1}{2}x + 1$, $x = 3$, and the x-axis has area $= \frac{1}{2}\left(\frac{5}{2}\right)(5)$

6. We seek k for which $2x^3 - 7x^2 - 12x + 6 = k$ has double roots.
By synthetic division:

	2	-7	-12	$6-k$
x	2	$2x-7$	$2x^2-7x-12$	$\underbrace{2x^3-7x^2-12x+6-k}_{\text{must be 0}}$
x	2	$4x-7$	$\underbrace{6x^2-14x-12}_{\text{also must be 0}}$	

$2(3x^2-7x-6) = 2(3x+2)(x-3) = 0$

When $x = -\frac{2}{3}$, $y = \frac{278}{27}$

When $x = 3$, $y = -39$