Minnesota State High School Mathematics League

Individual Event

2005-06 Event 1A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

5 ____ 1. Express as the quotient of two relatively prime integers the expression

$$\left[\frac{1}{12}+\frac{1}{15}+\frac{1}{20}\right]^{-1}$$

18.90 ____ 2. The shirt I was interested in was on a table with a sign, "Take 25% of the marked price." I was about to buy it when the clerk told me that the next day there was to be a sale that would advertise, "Take 1/3 off the already reduced price." If the shirt was originally marked \$37.80, how much will it cost the next day?

5292 ____ 3. Find the least common multiple of 108, 84, and 147.

4. The integer $N = 10100$ is expressed using base $b > 1$. Express N as a product of two integers, expressed as polynomials in b, that are both greater than 1.

$N = b^2(b^2+1)$

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1. $$\left[\frac{1}{3\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 5}\right]^{-1} = \left[\frac{5+4+3}{3\cdot 4\cdot 5}\right]^{-1}$$

$$\frac{3\cdot 4\cdot 5}{5+4+3} = \frac{3\cdot 4\cdot 5}{3\cdot 4} = 5$$

2. $$\frac{3}{4}(37.80) = 28.35$$

$$\frac{2}{3}(28.35) = 18.90$$

3. $2^2\cdot 3^3$, $2^2\cdot 3\cdot 7$, $3\cdot 7^2$

$$\text{lcm} = 2^2\cdot 3^3\cdot 7^2 = 5292$$

4. $$10100_b = b^4 + b^2$$

$$= b^2(b^2+1)$$

(Since b is an integer > 1, $b^2 > b$, so both factors are > 1.)

Minnesota State High School Mathematics League

Individual Event

2005-06 Event 1B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

90° 1. Two regular hexagons $ABCDEF$ and $DGHIJK$ (Figure 1) are positioned so that A, D, and I are collinear. If $KD = 2DE$, find the measure in degrees of $\angle KED$.

144° 2. In isosceles ΔABC (Figure 2), the trisectors of the angle at A meet the bisectors of the angles at B and C in points D and E. If $m(\angle BAC) = 36^\circ$, what is $m(\angle BDE)$?

$2a\sqrt{3}$ 3. In Figure 1, extend AF and IJ to meet at M. Given that $AF = a$, what (in terms of a) is the length of MK?

$\sqrt{7}$ 4. Again refer to Figure 2, but this time do not assume that $\angle A$ is trisected, or that the base angles are bisected. Rather, assume that ΔABC is equilateral, that ΔADE is isosceles with a base DE of length 1, and that both ΔADB and ΔAEC are isosceles. If $AB = 5$, how long is AD?

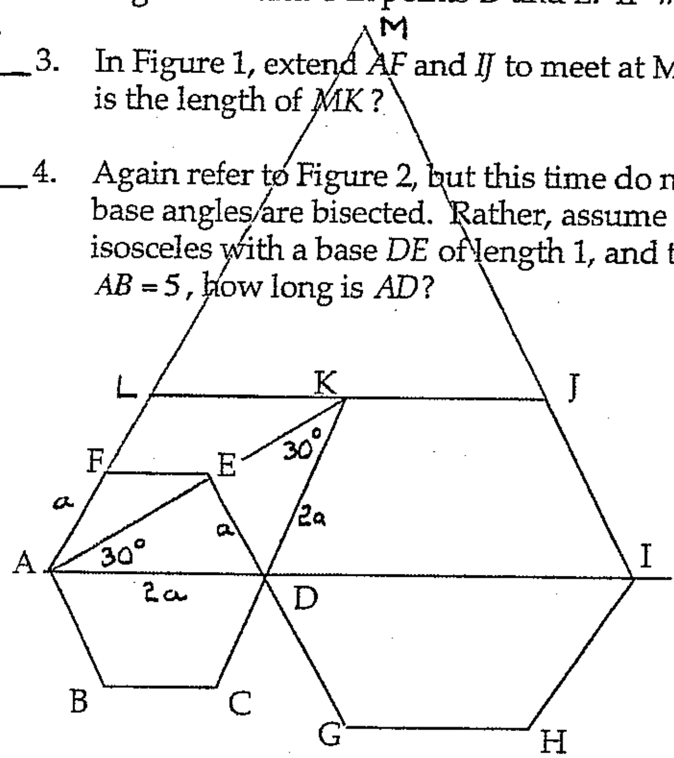


Figure 1

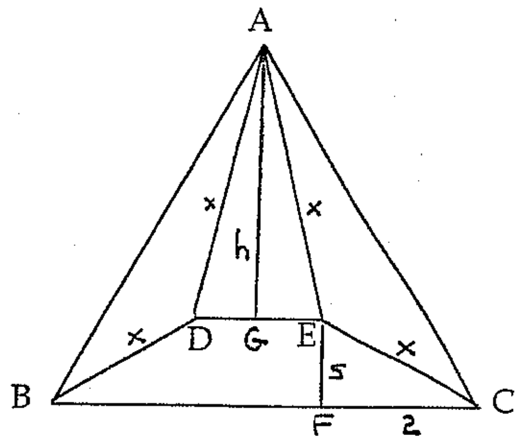


Figure 2

1. See labels on Figure 1. Since $AD = 2a$, ΔADE is a 30°-60°-90° Δ as is ΔKED. $\therefore \angle KED = 90^\circ$

2. $\angle DAB = 12^\circ$; $\angle DBC = \frac{1}{2}\left[\frac{1}{2}(180-36)\right] = 36^\circ$

$\angle BDE = \frac{1}{2}\left[360 - 2(36)\right] = 144^\circ$

3. ΔJLM is equilateral with sides of length $4a$. Its altitude is $MK = 2a\sqrt{3}$

4. Using the notation on the figure above, $h + s = \frac{5}{2}\sqrt{3}$ so $h = \frac{5}{2}\sqrt{3} - s$

From ΔCEF, $x^2 = s^2 + 4$

From ΔAGE, $x^2 = h^2 + \frac{1}{4}$

Subtracting, $0 = s^2 - h^2 + \frac{15}{4}$

Solve $s^2 - \left(\frac{5}{2}\sqrt{3} - s\right)^2 + \frac{15}{4} = 0$

$s = \sqrt{3}$; $x^2 = 3 + 4 = 7$

Minnesota State High School Mathematics League

Individual Event

2005-06 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{3}{4}$ 1. Figure 1 shows the graph of $y = a\sin bx$. What is b?

6 2. The units on the x and y axes in Figure 1 are the same, and a is an integer. What is it?

3.868 3. In isosceles ΔABC (Figure 3), the trisectors of the angle at A meet the bisectors of the angles at B and C in points D and E. A perpendicular is dropped from D to a point F on AB. If $m(\angle BAC) = 36°$ and $AB = 5$, how long is AF?

.827 4. Refer again to Figure 3 and the information given in Problem 3. How long is DE?

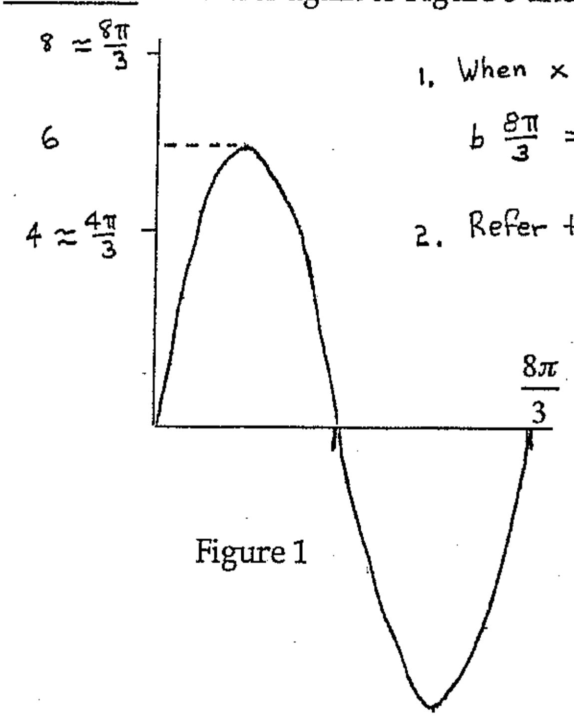


Figure 1

1. When $x = \frac{8\pi}{3}$, $bx = 2\pi$

$b\,\frac{8\pi}{3} = 2\pi$; $b = \frac{6\pi}{8\pi} = \frac{3}{4}$

2. Refer to Figure 1.

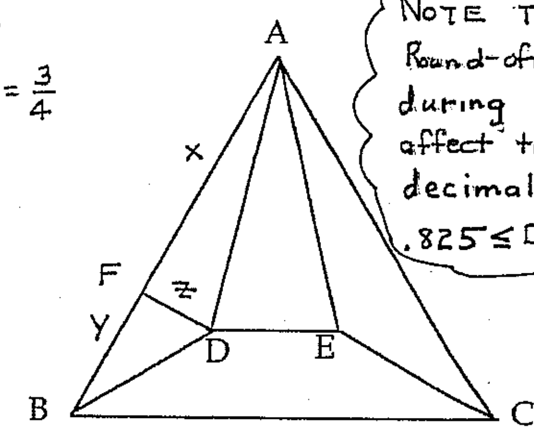


Figure 3

NOTE TO GRADERS
Round-off choices during solution will affect the third decimal. Accept $.825 \le DE \le .829$

3. As shown Event 1B, problem 3,

$\angle DBF = 36°$

$\tan 12° = \frac{z}{x}$ so $z = x(.2126)$

$\tan 36° = \frac{z}{5-x}$ so $z = (5-x)(.7265)$

Solve to find $x = 3.868$

4. From $\cos 12° = \frac{x}{AD}$ and problem 3,

$AD = \frac{3.868}{\cos 12°} = 3.954$

$\sin 6° = \frac{DE/2}{AD} = \frac{DE}{2\,AD}$

$DE = 2(3.954)\sin 6° = .827$

Minnesota State High School Mathematics League

Individual Event

2005-06 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{2+\sqrt{3}}{4}$, $\frac{-2-\sqrt{3}}{4}$ 1. Find the roots of $(2x+1)^2 = \frac{3}{4}$

65 2. What will be the remainder if $2x^5 - 3x^4 - 4x^3 - 5x^2 - 6x - 7$ is divided by $x-3$?

$\left(-\frac{4}{3}, -\frac{5}{2}\right)$ 3. Find the coordinates (both of them) of the lowest point on the graph of $9x^2 + 24x - 72y - 164 = 0$.

4. The coefficients of $z^3 + az^2 + bz + c = 0$ are all real numbers. The three roots z_1, z_2, and z_3 of the equation satisfy

$$z_1 z_2 z_3 = 1 \qquad \text{and} \qquad z_1 + z_2 + z_3 = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}$$

For what choices of a will there be some non-real complex roots?

$-3 < a < 1$

1. $2x + 1 = \frac{\pm\sqrt{3}}{2}$

$2x = -1 \pm \frac{\sqrt{3}}{2} = \frac{-2 \pm \sqrt{3}}{2}$

2. $\frac{p(x)}{x-3} = p(3) = 65$

OR by synthetic division

	2	−3	−4	−5	−6	−7	
3	2	3	5	10	24		65

3. $9\left(x^2 + \frac{8}{3}x + \frac{16}{9}\right) = 72y + 164 + 16$

$9\left(x + \frac{4}{3}\right)^2 = 72\left(y + \frac{5}{2}\right)$

$\left(x + \frac{4}{3}\right)^2 = 8\left(y + \frac{5}{2}\right)$

Min occurs at $\left(-\frac{4}{3}, -\frac{5}{2}\right)$

4. Immediately, we have $c = -1$

Multiply the second expression by $1 = z_1 z_2 z_3$ to get

$$\underbrace{z_1 + z_2 + z_3}_{-a} = \underbrace{z_2 z_3 + z_1 z_3 + z_1 z_2}_{b}$$

The equation is

$z^3 + az^2 - az - 1 = 0$

	1	a	$-a$	−1	
1	1	$a+1$	1		0

One root is 1. The other roots are found from

$$z = \frac{-(1+a) \pm \sqrt{(1+a)^2 - 4}}{2}$$

These roots will be complex if $(1+a)^2 - 4 < 0$; $-3 < a < 1$

Minnesota State High School Mathematics League

Team Event

2005-06 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

$\frac{abc}{a+b+c}$ 1. Express $\left[\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right]^{-1}$ as a single quotient.

2. The integer $M = 100011$ is expressed using base $b > 1$. Express M as a product of two integers, expressed as polynomials in b, that are both greater than 1.

$(b^2+b+1)(b^3-b^2+1)$

3. Find to the nearest tenth of a degree all angles θ, $0 \le \theta < 360$, that satisfy

$$15\tan^2\theta - \tan\theta\sec\theta - 6\sec^2\theta = 0$$

$41.8°$, $138.2°$, $216.9°$, $323.1°$ (1 point for each correct answer)

4. In ΔABC, $\angle A = 40°$, $\angle B = 25°$, and $AC = 3$. What is the area of ΔABC?

6.203

5. For distinct real numbers a, b, and c, division of the polynomial $p(x)$ by $x-a$ leaves a remainder of a; division of the polynomial $p(x)$ by $x-b$ leaves a remainder of b; division of the polynomial $p(x)$ by $x-c$ leaves a remainder of c. What is the remainder when $p(x)$ is divided by $(x-a)(x-b)(x-c)$? Hint: rephrase the question. Let $p(x) = (x-a)(x-b)(x-c)q(x) + r(x)$. Find $r(x)$. in simplest form.

$r(x) = x$

6. Given: $x + y + z = 0$; $x^2 + y^2 + z^2 = 36$; $x^3 + y^3 + z^3 = 105$

5 One of the unknowns, say x, (by symmetry, it could be any one of them) is real; the other two are complex. Find x.

The ideas for problems 2, 5, and 6 came from a source to be acknowledged at the end of the season.

Team ______________________

Team Event 1 Solutions

1. $\left[\dfrac{c}{abc}+\dfrac{a}{abc}+\dfrac{b}{abc}\right]^{-1}$

$= \dfrac{abc}{a+b+c}$

2. $100011_b = b^5+b+1$

To factor, assume factors of

$(b^2+mb+1)(b^3+nb^2+pb+1)$

Multiplication and simplification give

$b^5+\underbrace{(m+n)}_{0}b^4+\underbrace{(p+mn+1)}_{0}b^3$

$+\underbrace{(1+mp+n)}_{0}b^2+\underbrace{(m+p)}_{1}b+1$

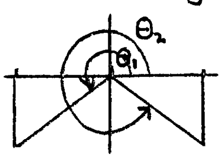
Solution gives $m=1,\ n=-1,\ p=0$

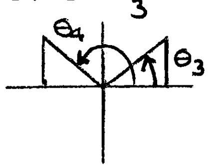
3. $15\dfrac{\sin^2\theta}{\cos^2\theta}-\dfrac{\sin\theta}{\cos^2\theta}-\dfrac{6}{\cos^2\theta}=0$

$15\sin^2\theta-\sin\theta-6=0$

$(5\sin\theta+3)(3\sin\theta-2)=0$

$\sin\theta=-\frac{3}{5}$ $\qquad$ $\sin\theta=\frac{2}{3}$





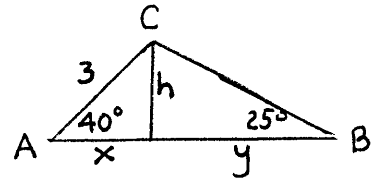
$\theta_2=360+\text{Arcsin}\left(-\frac{3}{5}\right)$
$=323.1$

$\theta_1=180+\text{Arcsin}\ \frac{3}{5}$
$=216.9$

$\theta_3=\text{Arcsin}\ \frac{2}{3}$
$=41.8^\circ$

$\theta_4=180-41.8=138.2$

4.



$\sin 40^\circ=\frac{h}{3}$

$h=3\sin 40^\circ=1.9284$

$\cos 40^\circ=\frac{x}{3}$

$x=3\cos 40^\circ=2.2981$

$\tan 25^\circ=\frac{h}{y}$

$y=\dfrac{1.9284}{\tan 25}=4.1355$

$x+y=6.4336$

$\text{Area}=\frac{1}{2}(6.4336)(1.9284)=6.203$

5. By the remainder theorem

$p(a)=a;\quad p(b)=b;\quad p(c)=c$

Form $P(x)=p(x)-x$. Since $P(x)$ has three distinct zeroes a,b,c, $\deg P(x)=\deg p(x)\geq 3$. Let

$p(x)=(x-a)(x-b)(x-c)q(x)+r(x)$

where $\deg r(x)\leq 2$

$\left.\begin{array}{l} p(a)=r(a)=a\\ p(b)=r(b)=b\\ p(c)=r(c)=c\end{array}\right\}\Rightarrow r(x)-x$

has three zeroes. Since $\deg(r(x)-x)\leq 2$, $r(x)-x\equiv 0$

$\boxed{r(x)=x}$

6. Let $t^3-at^2+bt-c=0$ have x, y, and z as roots. Then

$a=x+y+z=0;\quad b=xy+xz+yz;\quad c=xyz$

$(x+y+z)^2=x^2+y^2+z^2+2(xy+xz+yz)$; i.e. $0^2=36+2b$; $b=-18$

The equation is now seen to be $t^3-18t-c=0$. Since x, y, and z are solutions,

$$\text{Add}\begin{cases}x^3-18x-c=0\\ y^3-18y-c=0\\ z^3-18z-c=0\end{cases}$$

$105-18(x+y+z)-3c\Rightarrow c=35$

Solve $t^3-18t+35=0$

$$\begin{array}{r|rrrr|} & 1 & 0 & -18 & 35\\ \hline \rightarrow 5 & 1 & 5 & 7 & 0\end{array}$$

$t^2+5t+7=0$ has complex roots