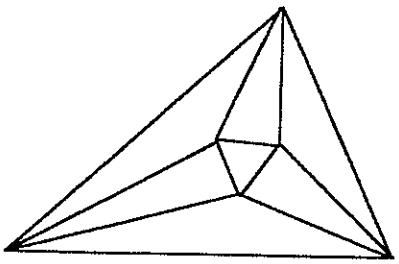
Solutions



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 1A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

Write the expressions in problems 1 and 2 as the quotient of two relatively prime integers.

$\frac{3}{11}$ 1. .272727 .... =

$\frac{33}{40}$ 2. $\dfrac{\frac{1}{6}+\frac{3}{4}}{\frac{4}{9}+\frac{2}{3}} =$

Accept 17,784 (in case someone knows rounding to the nearest dollar is permitted)

17,783.87 3. Carrie A. Handful found on line 14 of her 2003 Minnesota Income Tax that her taxable income was \$243,812. She then read, "If line 14 is over \$110,390, enter on line 15 \$7310.24 + 7.85% of the amount over \$110,390." How much should she enter on line 15?

4032 4. The set of three positive integers $\{12, 45, m\}$ has a greatest common divisor $d > 1$ and a least common multiple of 2520. What is the sum of the possible values for the integer m?

1. Let $x = .272727\ldots$

$100x = 27.2727\ldots = 27 + x$

$99x = 27$

$x = \frac{3 \cdot 9}{11 \cdot 9} = \frac{3}{11}$

2. $\dfrac{\left(\frac{1}{3\cdot 2} + \frac{3}{2\cdot 2}\right)\frac{2\cdot 2\cdot 3\cdot 3}{}}{\left(\frac{4}{3\cdot 3} + \frac{2}{3}\right)\frac{2\cdot 2\cdot 3\cdot 3}{}} = \frac{6+27}{16+24}$

$= \frac{33}{40}$

Multiply numerator and denominator by the l.c.d. of the four fractions

3.
$$\begin{array}{r} 243,812 \\ 110,390 \\ \hline 133422 \\ \times .0785 \\ \hline 10,473.63 \\ 7310.24 \\ \hline 17,783.87 \end{array}$$

4. $\{2\cdot 2\cdot 3,\ 3\cdot 3\cdot 5,\ m\}$

Since the gcd > 1, it must be 3. Since $\text{lcm} = 2^3 \cdot 3^2 \cdot 5 \cdot 7$, m must have factors of $3, 2^3$, and 7. It might have as additional factors 3 and 5. Possible:

$168 = 168$

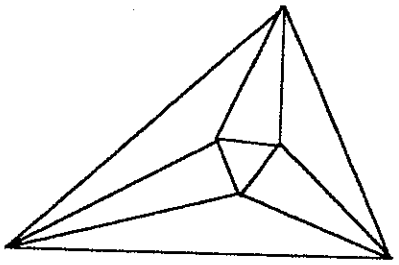
$168 \cdot 3 = 504$

$168 \cdot 5 = 840$

$168 \cdot 3 \cdot 5 = 2520$

4032

Solutions



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 1B

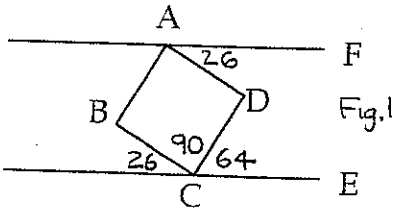
The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

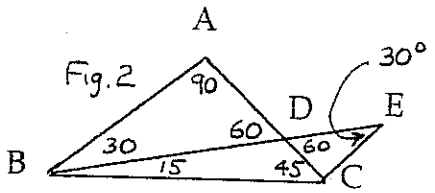
26° 1. Parallel lines pass through the vertices A and C of square $ABCD$ as shown in Figure 1. If $\angle ECD = 64°$, what is the measure of $\angle FAD$?

30° 2. Isosceles ΔABC has as its apex $\angle A = 90°$. BD is drawn so that $\angle CBD = \frac{1}{3}\angle CBA$. Line CE is drawn perpendicular to AC and meets the extension of BD at E (Figure 2). What is the measure of $\angle BEC$?

$\frac{\alpha+\beta}{2}$ 3. In ΔABC with $AC < BC$, let the measure of the angles at A and B be α and β respectively. Extend CA to D so that $CD = CB$ (Figure 3). Express the measure of $\angle CDB$ in terms of α and β.

4. In isosceles ΔABC with base BC and D a point on leg AC, $BC = BD = DA$. What is the measure of $\angle BAC$?





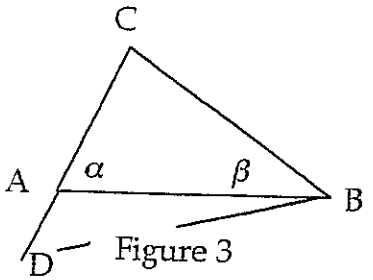


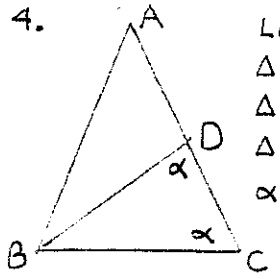
Figure 3

$\angle C = 180 - (\alpha + \beta)$

$\angle CDB = \angle CBD$

$= \frac{180 - \angle C}{2} = \frac{\alpha+\beta}{2}$

4.



Let $\alpha = \angle BCD$

ΔBCD isosceles $\Rightarrow \angle CBD = 180 - 2\alpha$

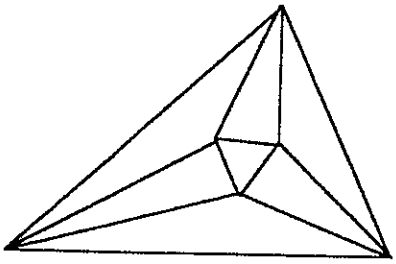
ΔABC isosceles $\Rightarrow \angle A = 180 - 2\alpha$

ΔBDA isosceles $\Rightarrow \angle ABD = 180 - 2\alpha$

$\alpha = \angle ABD + \angle CBD = 2(180 - 2\alpha)$

$5\alpha = 360$; $\alpha = 72$; $\angle A = 180 - 144 = 36$

Solutions



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{\sqrt{3}}{2}$ 1. Express in exact, simplified form the value of $\cos\frac{5\pi}{6}+\tan\frac{\pi}{3}$.

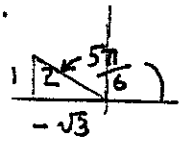
$\theta+\pi$ 2. The line from the origin O to the point $A(-4,-2)$ forms with the positive x-axis an angle α that is between π and $\frac{3\pi}{2}$. If $\csc\theta=\sqrt{5}, 0<\theta<\frac{\pi}{2}$, express α as a radian expression involving θ.

$\frac{16\sqrt{3}}{3}$ 3. Two tangents to a circle, each of length 12, intersect to form an angle of 60°. If the tangents meet the circle at points A and B, the length of the major arc AB is $k\pi$. Give the exact, rationalized value of k.

4. Hipparchus (190-120 B.C.) estimated the distance from the surface of the earth to the moon by the following method, described here using miles as the unit of measurement. Using an eclipse of the moon as the signal for obtaining simultaneous observations, it was found that when observers were separated by 6220 miles (measure in the usual way along the surface of the earth), observer A could see the moon M just over the horizon while observer B saw it directly overhead (Figure 1). Estimates of the radius of the earth at the time were 4000 miles. Using these numbers, what is BM to the nearest mile?

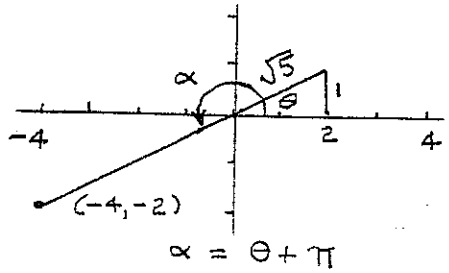
249,234

1.



$-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{1}=\frac{\sqrt{3}}{2}$

2.



$\alpha=\theta+\pi$

4.

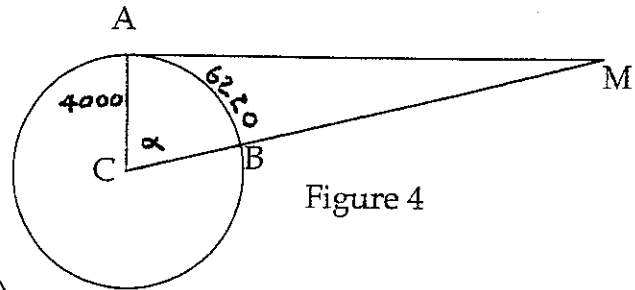
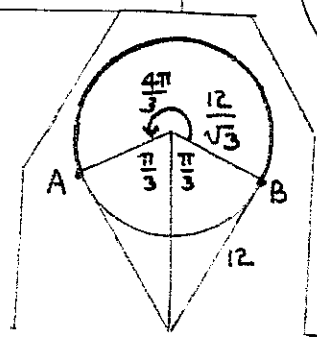


Figure 4

In radians, $\alpha=\frac{6220}{4000}=1.555$

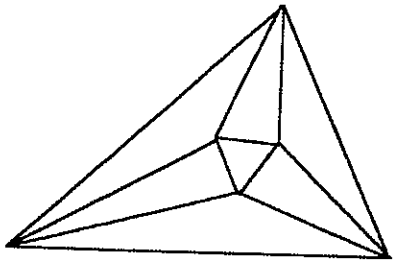
$\cos\alpha=\frac{4000}{CM}$, so $CM=\frac{4000}{\cos(1.555)}$

$BM=CM-4000=253,234-4000$



3. $\widehat{AB}=\frac{4\pi}{3}\cdot\frac{12}{\sqrt{3}}=\frac{16\pi}{\sqrt{3}}=\frac{16\sqrt{3}}{3}\pi$

Solutions



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

No calculators on this event.

1 1. What is the product of the roots of $2x^2 - 3x + 2 = 0$?

$b = \pm 12$ 2. Find b so that the equation $4x^2 + bx + 9 = 0$ has one rational (double) root. (contributed at the 2003 coaches conference)

Give full credit for $b = 12$ or for $b = -12$

1/3 3. Given that -2 is one root of $6x^3 - 5x^2 - 29x + 10 = 0$, find the other two roots

5/2

4. The polynomial $3x^3 - 13x^2 + ax + b = 0$, a and b real numbers, has $2 - i$ as one of its roots. Where does the graph of $y = 3x^3 - 13x^2 + ax + b$ cross the x-axis?

1/3

Accept $(\frac{1}{3}, 0)$

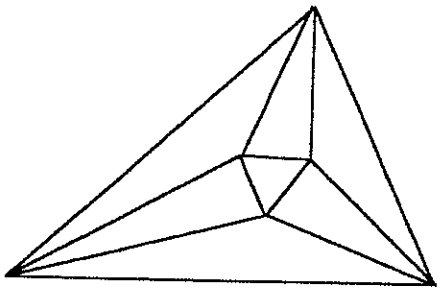
1. Thm The product of the roots of $1x^2 + bx + c = 0$ is c. The product of the roots of $x^2 - \frac{3}{2}x + 1 = 0$ is 1.

2. The roots are equal when the discriminant = 0.
$b^2 - 4(4)(9) = 0$; $b = \pm 12$

3. Divide by $x+2$, synthetically or otherwise to get
$6x^3 - 5x^2 - 29x + 10 = (x+2)(6x^2 - 17x + 5)$
Then $6x^2 - 17x + 5 = (3x - 1)(2x - 5)$

4. The sum of the roots of
$x^3 - \frac{13}{3}x^2 + \frac{a}{3}x + \frac{b}{3} = 0$
is $-(-\frac{13}{3})$. Let the roots be $2 - i$, $2 + i$, and r.
$(2 - i) + (2 + i) + r = \frac{13}{3}$
$r = \frac{13}{3} - 4 = \frac{1}{3}$

Answers



Minnesota State High School Mathematics League

Team Event

2004-05 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

$\sqrt{3}-1$ or .732 1. Isosceles ΔABC has as its apex $\angle A = 90°$. BD is drawn so that $\angle CBD = \frac{1}{3}\angle CBA$, and it is extended to meet a line erected at C, perpendicular to AC at E (Figure 1). If the legs of the triangle have length 1, what is the length of CE?

90° 2. Isosceles ΔABC has as its apex $\angle A = 90°$. Adjacent trisectors of its angles meet to form ΔDEF as shown in Figure 2. What (in degrees) is the measure of $\angle DEC$?

$-3 \le x \le 1$ 3. For what value(s) of x is $f(x) = |-3-x| + |1-x|$ a minimum?

.766 4. An observer on a boat B traveling due east notices a light 20° to the northeast. After traveling 1 mile, the light is 40° to the northeast (Figure 4). Find the distance x in miles that the boat will have to traveled when the light L is due north.

$b = \frac{2}{3}$ 5. The equation $9x^2 + 12x = 32$ can be written in the form $(x+b)^2 = k^2$. Find (i) b; (ii) k.
$k = 2$ (2 points for each correct answer)

1 6. Give a numeric value for the continued fraction $\cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{2+\ddots}}}}$

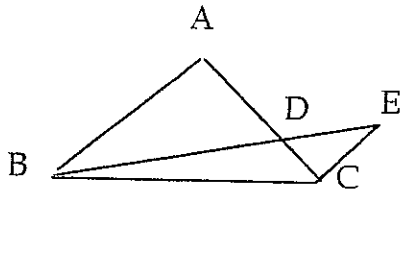


Figure 1

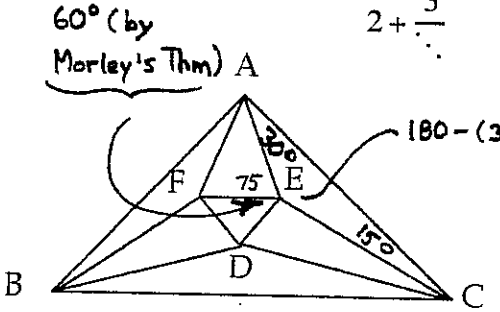


Figure 2

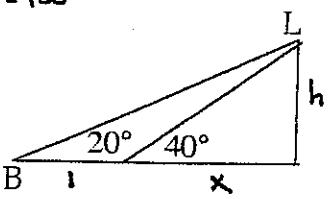
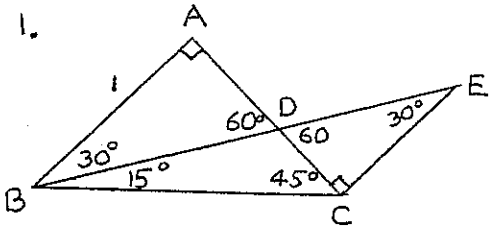


Figure 4

Team_____

Team Event 1 Solutions

1.



1. Label the angles as they were determined in Prob 2 of Event B. Then

$CD = 1 - AD = 1 - \frac{1}{\sqrt{3}}$

$CE = \sqrt{3}\,CD = \sqrt{3} - 1$

2. See Figure 2 on the Answer Sheet.

$\angle DEC + 135 + 75 + 60 = 360$

$\angle DEC = 90°$

4. See Figure 4 on the Answer Sheet

$\tan 20° = \frac{h}{x+1}$ so $h - x\tan 20 = \tan 20$

$\tan 40° = \frac{h}{x}$ so $h - x\tan 40 = 0$

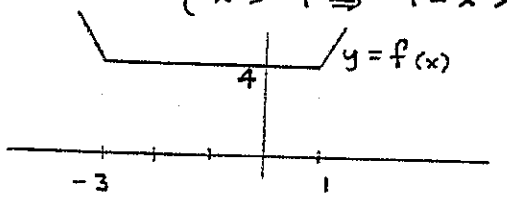
$x\tan 40 - x\tan 20 = \tan 20$

$x = \frac{\tan 20}{\tan 40 - \tan 20} = .766$

3. $(-\infty, -3]$ $\left\{ \begin{array}{l} x \le -3 \Rightarrow -x-3 > 0 \\ x < 1 \Rightarrow 1-x > 0 \end{array} \right\}$ $f(x) = (-3-x) + (1-x) = -2-2x$

$[-3, +1]$ $\left\{ \begin{array}{l} x \ge -3 \Rightarrow -x-3 < 0 \\ x \le 1 \Rightarrow 1-x > 0 \end{array} \right\}$ $f(x) = -(-3-x) + (1-x) = 4$

$[1, \infty)$ $\left\{ \begin{array}{l} x > -3 \Rightarrow -x-3 < 0 \\ x > 1 \Rightarrow 1-x > 0 \end{array} \right\}$ $f(x) = -(-3-x) - (1-x) = 2+2x$



f achieves its minimum of 4 for x in $[-3, 1]$; i.e. $-3 \le x \le 1$.

5. $9(x^2 + \frac{12}{9}x + \frac{4}{9}) = 32 + 4 = 36$ ($\frac{4}{3}$)

$\left(x + \frac{2}{3}\right)^2 = \frac{36}{9} = 4$

6. Set the given fraction equal to x. Then

$$x = \frac{3}{2+x}$$

$x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

Clearly, $x > 0$, so $x = 1$