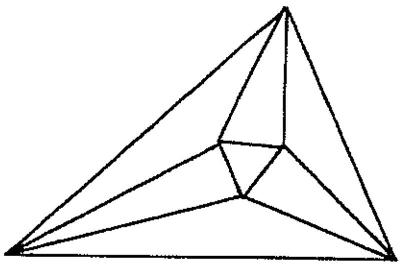
Solutions



# Minnesota State High School Mathematics League

## Individual Event

## 2003-04 Event 1A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{14}{11}$ 1. The decimal 1.545454 . . . can be expressed as $\frac{17}{11}$, the quotient of two relatively prime integers. Express 1.272727. . . as the quotient of two relatively prime integers.

39.1% 2. When a group of Mathleaguers went to a Twins game last summer, we got with our tickets a brochure saying that with each ticket we could buy a coupon for \$3.50 that would be good for a hot dog and a soft drink. This, it said, would be a saving of more than 30% off the regular price. The regular price for a hot dog and soft drink was \$5.75. What (to the nearest tenth of a percent) was the actual saving?

3. Let $0 < a < \frac{1}{4} < \frac{3}{4} < b < 1$. Use a similar string of inequalities to order from smallest to largest $\sqrt{a}$, $a^2$, $\frac{1}{a}$, $\sqrt{b}$, $b^2$, $\frac{1}{b}$,

$a^2 < \sqrt{a} < b^2 < \sqrt{b} < \frac{1}{b} < \frac{1}{a}$

540 4. The set of three positive integers {15, 25, $k$} has a greatest common divisor of 5 and a least common multiple of 450. What is the sum of the possible values for the integer $k$?

1. Set $x = .272727...$

$100x = 27 + x$

$x = \frac{27}{99} = \frac{3}{11}$

$1 + x = \frac{11+3}{11} = \frac{14}{11}$

2. If $p$ is the actual percent saved

$p(5.75) = 2.25$

$p = \frac{225}{575} = .391$

3. $a < \frac{1}{4} \Rightarrow \sqrt{a} < \frac{1}{2}$ and $b > \frac{3}{4} \Rightarrow b^2 > \frac{9}{16} > \frac{1}{2}$

$a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$ and $b < 1 \Rightarrow \frac{1}{b} > 1$

$\therefore \quad a^2 < \sqrt{a} < b^2 < \sqrt{b} < \frac{1}{b} < \frac{1}{a}$

4. $\text{lcm} = 450 = 3^2 \cdot 5^2 \cdot 2$.

$k$ must have one factor of 2, two factors of 3; $k$ must have one factor of 5 (since gcd = 5), and it might have two.

$k = 2 \cdot 3^2 \cdot 5$ or $k = 2 \cdot 3^2 \cdot 5^2$

sum $= 2 \cdot 3^2 \cdot 5 + 2 \cdot 3^2 \cdot 5^2 = 540$

Solutions

# Minnesota State High School Mathematics League

## Individual Event

## 2003-04 Event 1B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

_________ 1. Figure 1 shows an equilateral $\Delta RST$ inscribed in a regular pentagon $ABCDE$ in such a way that $R$ is the midpoint of $\overline{CD}$ and $\overline{ST} \parallel \overline{CD}$. What is the measure of $\angle ATS$?

_________ 2. In Figure 1, what is the measure of $\angle ESR$?

_________ 3. In an isosceles $\Delta ABC$ with $AC=BC$, let $\overline{CD}$ and $\overline{CE}$ be the angle trisectors of $\angle C$, and let $\overline{AF}$ and $\overline{AG}$ be the angle trisectors of $\angle A$ (Figure 3). Let $\overline{AF}$ intersect $\overline{CD}$ at $K$ and $\overline{CE}$ at $L$. If $\angle A = 54°$, what will be the measure of $\angle KLE$?

_________ 4. If in Figure 3, we also draw the angle trisectors of $\angle B$ and let $R$, $S$, and $T$ be the points where pairs from adjacent vertices intersect (as in the logo at the top of the page), what will be the measure of $\angle ART$?

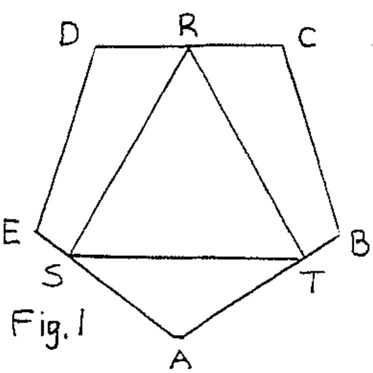



Fig. 1

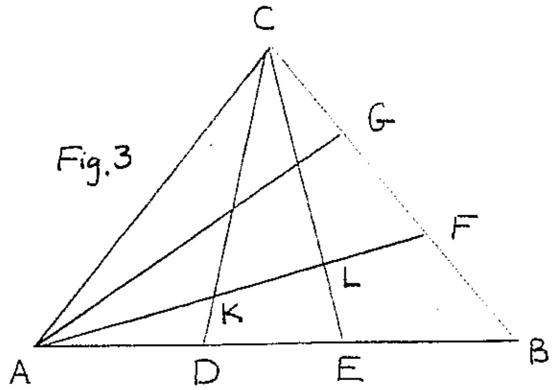
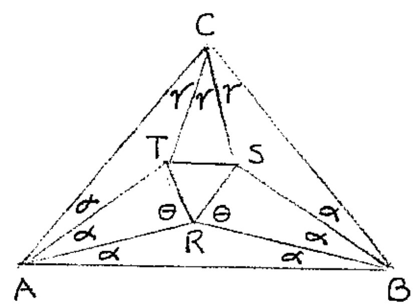



Fig. 3




1. $\angle A = \dfrac{3(180)}{5} = 108°$

$\angle ATS = \frac{1}{2}[180 - 108] = 36°$

2. $\angle ESR = 180 - (36 + 60) = 84°$

3. Let $\alpha = \frac{1}{3}\angle A = 18°$, and let

$\gamma = \frac{1}{3}\angle C = \frac{1}{3}[180 - 2(54)] = 24°$

3. (continued)

$\angle AFG = \alpha + \angle B = 18 + 54 = 72°$

From $\Delta LFC$, $\angle CLF = 180 - (72 + \gamma) = 84°$

$\therefore \angle KLE = 84°$

4. $\angle ARB = 180 - (\alpha + \alpha) = 180 - 36 = 144$

$\Delta CBS \cong \Delta ACT$ (ASA), so $\Delta ART \cong \Delta BRS$ (SSS)

Let $\theta = \angle ART = \angle BRS$. Morley's Thm $\Rightarrow$

$\angle TRS = 60$. $\therefore 2\theta + 60 + 144 = 360$; $\theta = 78°$

Solutions

# Minnesota State High School Mathematics League

## Individual Event

## 2003-04 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

$\frac{2}{\sqrt{5}}$ or $\frac{2\sqrt{5}}{5}$ 1. What is the sine of the second quadrant angle $\alpha$ shown in Figure 1?

$\frac{-1}{\sqrt{5}}$ or $\frac{-\sqrt{5}}{5}$ 2. For the angle $\alpha$ shown in Figure 1, what is $\sin\left(\alpha+\frac{\pi}{2}\right)$?

3. Express $\sqrt{\frac{1-\sin x}{1+\sin x}}$ as the difference of two trigonometric functions (such as $\sin x - \csc x$), given that $x$ is in the second quadrant.

$\tan x - \sec x$

4. Two tangents to a circle, each of length 12, intersect to form an angle of 30°. If the tangents meet the circle at points A and B, what is the length, accurate to three places to the right of the decimal, of the long arc $\widehat{AB}$?

11.785

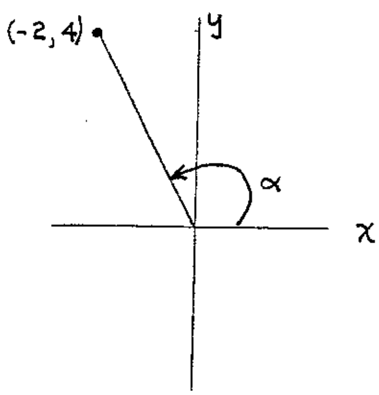



Figure 1

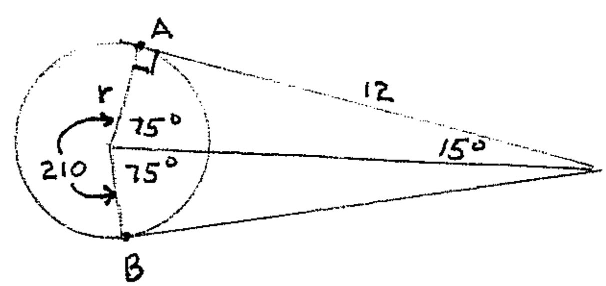
1. $\sin\alpha = \frac{4}{\sqrt{4+16}} = \frac{2}{\sqrt{5}}$

2. $\sin(\alpha + \frac{\pi}{2}) = \cos\alpha = \frac{-2}{2\sqrt{5}}$

3. $\sqrt{\frac{(1-\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)}} = \frac{1-\sin x}{\sqrt{\cos^2 x}}$

$= \frac{1-\sin x}{-\cos x} = -\sec x + \tan x$

4.




$\tan 15° = \frac{r}{12}$, so $r = 12\tan 15°$

$\widehat{AB} = \frac{210}{360}(2\pi r) = \frac{7}{12}(2\pi)12\tan 15°$

$= 11.785$

Solutions

# Minnesota State High School Mathematics League

## Individual Event

### 2003-04 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

**All questions in this event refer to the polynomial** $p(x) = 2x^2 + 2x + 3$

$\frac{3}{2}$ 1. What is the product of the roots of $p(x) = 0$?

$A(\frac{1}{2}, \frac{9}{2})$ $B(-2, 7)$ 2. The graph of $y = p(x)$ intersects the graph of $y = 5 - x$ in two points A and B. Give the coordinates (both of them) of both points.

Of course these can be reversed: $A(-2,7)$, $B(\frac{1}{2}, \frac{9}{2})$

3. Write $y = p(x)$ in the form $y - k = 4a(x-h)^2$.

$y - \frac{5}{2} = 4(\frac{1}{2})(x + \frac{1}{2})^2$

4. The equation $6x^4 + 10x^3 + 15x^2 + 8x + 3 = 0$ has two of the same roots as the equation $p(x) = 0$. For what second degree polynomial $r(x)$ does $r(x) = 0$ have roots equal to the other two roots of the given 4th degree polynomial equation?

$r(x) = 3x^2 + 2x + 1$

1. $x^2 + x + \frac{3}{2} = 0$

The product of the roots is $\frac{3}{2}$

2. $2x^2 + 2x + 3 = 5 - x$

$2x^2 + 3x - 2 = 0$

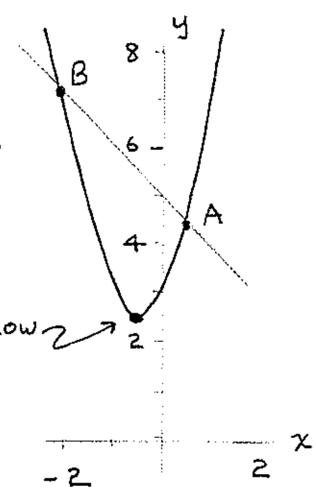
$(2x - 1)(x + 2) = 0$

$A(\frac{1}{2}, \frac{9}{2})$; $B(-2, 7)$

3. $y = 2(x^2 + x + \frac{1}{4}) + 3 - \frac{1}{2}$

$= 2(x + \frac{1}{2})^2 + \frac{5}{2}$

The lowest point is $(-\frac{1}{2}, \frac{5}{2})$




4.

$$\begin{array}{r}
3x^2 + 2x + 1 \\
2x^2 + 2x + 3 \,\big)\, 6x^4 + 10x^3 + 15x^2 + 8x + 3 \\
6x^4 + 6x^3 + 9x^2 \qquad\quad \\
\hline
4x^3 + 6x^2 + 8x \quad \\
4x^3 + 4x^2 + 6x \quad \\
\hline
2x^2 + 2x + 3 \\
2x^2 + 2x + 3 \\
\hline
\end{array}$$

$r(x) = 3x^2 + 2x + 1$

Answers

# Minnesota State High School Mathematics League

## Team Event

## 2003-04 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

14,976 1. The set of four positive integers $\{42, 54, 60, k\}$ has a greatest common divisor of 6 and a least common multiple of 7560. What is the sum of the possible values for the integer $k$?

2. Figure 2 shows a circle of radius $r$ with a central angle $\theta$, $0 \le \theta \le \frac{\pi}{2}$. The inequalities $Area(\Delta OBD) < Area(\text{sector } OBD) < Area(\Delta OBC)$ can be used to show $f(\theta) < \theta < g(\theta)$ where $f(\theta)$ and $g(\theta)$ represent trigonometric functions. Write the inequality.

$\sin\theta < \theta < \tan\theta$

3. Describe a good calculator window in which one can see a continuous part of the graph of $y = P(x) = 30x^3 - 121x^2 + 162x - 72$ that shows all the roots of $P(x) = 0$.

xmin= ________ xmax= ________ ymin= ________ ymax= ________

Graders – This is intended to be an easy problem, rewarding those who use their calculator intelligently. See the solution sheet.

2 4. Give a numeric value for the continued fraction
$$\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{\ddots}}}}}$$

5. Figure 5 shows the graph of a parabola $y = x^2 + bx + c$ having its lowest point at $(m,n)$, $n < 0$. Express the roots of $x^2 + bx + c = 0$ in terms of $m$ and $n$.

$m \pm \sqrt{-n}$

6. Let $f(x) = 3x^2 - 2(a+b+c)x + (ab+ac+bc)$. Then $f\left(\frac{a+b}{2}\right)$ can be expressed as a rational number times the square of a term involving $a$ and $b$. Do so.

$-\frac{1}{4}(a-b)^2$

Team ____________________

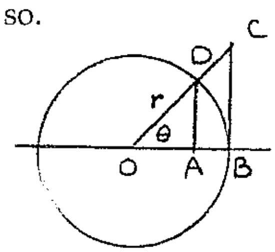



Figure 2

# Team Event 1 Solutions

1. $\{2\cdot 3\cdot 7,\ 2\cdot 3^3,\ 2^2\cdot 3\cdot 5,\ k\}$

has $\begin{cases} \text{g.c.d} = 2\cdot 3 \\ \text{l.c.m} = 2^3\cdot 3^3\cdot 5\cdot 7 \end{cases}$

g.c.d $= 6 \Rightarrow k$ has factors of $2, 3$

l.c.m of the given numbers is $2^2\cdot 3^3\cdot 5\cdot 7$. $\therefore$ $k$ has a factor of $2^3$.

Possible values for $k$ are

| | | |
|---|---|---|
| $2^3\cdot 3$ | $2^3\cdot 3^2$ | $2^3\cdot 3^3$ |
| $2^3\cdot 3\cdot 5$ | $2^3\cdot 3^2\cdot 5$ | $2^3\cdot 3^3\cdot 5$ |
| $2^3\cdot 3\cdot 7$ | $2^3\cdot 3^2\cdot 7$ | $2^3\cdot 3^3\cdot 7$ |
| $2^3\cdot 3\cdot 5\cdot 7$ | $2^3\cdot 3^2\cdot 5\cdot 7$ | $2^3\cdot 3^3\cdot 5\cdot 7$ |

$$\begin{aligned} \text{Sum} &= 2^3\cdot 3[1+5+7+35] \\ &\quad + 2^3\cdot 3^2[1+5+7+35] + 2^3\cdot 3^3[1+5+7+35] \\ &= (2^3\cdot 3 + 2^3\cdot 3^2 + 2^3\cdot 3^3)[48] \\ &= 14{,}976 \end{aligned}$$

4. Let the given expression be $x$. Then

$$x = \frac{6}{1+x}; \qquad x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0; \qquad x = 2 \text{ or } x = -3$$

But clearly $x > 0$, so $x = 2$.

5. The equation in standard form is

$$4a(y-n) = (x-m)^2$$

$$y = n + \frac{1}{4a}(x^2 - 2mx + m^2)$$

Since the coefficient of $x^2$ in the given equation is 1, $4a = 1$ and the standard form is

$$y - n = (x-m)^2$$

When $y = 0$, $x - m = \pm\sqrt{-n}$

$$x = m \pm \sqrt{-n} \qquad (n<0)$$

2. Area $(\Delta OAD)$ < Area (sector OBD) < Area $(\Delta OBC)$

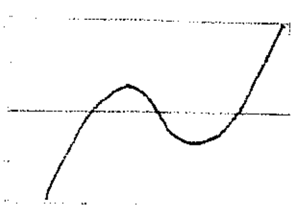
$$\frac{1}{2}(OB)(DA) < \frac{\theta}{2}r^2 < \frac{1}{2}(OB)(BC)$$

$OB = r$, $DA = r\sin\theta$, $BC = r\tan\theta$

$$\therefore\ \cancel{r^2}\sin\theta < \cancel{r^2}\,\theta < \cancel{r^2}\tan\theta$$

3. Enter the parameters submitted into a calculator. Give credit if you see (1) three roots between 1 and 1.6 (actually occur at $\frac{6}{5}, \frac{4}{3}, \frac{3}{2}$), and (2) a rise above the $x$ axis between $6/5$ and $4/3$, and (3) below the $x$-axis between $4/3$ and $3/2$. The window $1.0 \le x \le 1.6$, $-0.2 \le y \le 0.2$ (below) works well.

xmin must be $< 1.2$
xmax must be $> 1.5$
ymin must be $< -.05$
ymax must be $> .04$



6. $$f\left(\frac{a+b}{2}\right) = 3\left(\frac{a+b}{2}\right)^2 - \cancel{2}(a+b+c)\frac{a+b}{\cancel{2}} + ab + ac + bc$$

$$= \frac{3}{4}(a+b)^2 - (a+b)^2 - \cancel{ac} - \cancel{bc} + ab + \cancel{ac} + \cancel{bc}$$

$$= -\frac{1}{4}(a^2 + 2ab + b^2) + ab = -\frac{1}{4}(a^2 - 2ab + b^2)$$

$$= -\frac{1}{4}(a-b)^2$$